SIAM J. DISCRETE MATH.
Vol. 18, No. 4, pp. 794–824
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ON THE BEHAVIOR OF A FAMILY OF META-FIBONACCI SEQUENCES*

JOSEPH CALLAGHAN†, JOHN J. CHEW III†, AND STEPHEN M. TANNY†

Abstract. A family of meta-Fibonacci sequences is defined by the k-term recursion

$$T_{a,k}(n) := \sum_{i=0}^{k-1} T_{a,k}(n-i-a-T_{a,k}(n-i-1)), \quad n > a+k,\ k \geq 2,$$

with initial conditions $T_{a,k}(n) = 1$ for $1 \leq n \leq a+k$. Some partial results are obtained for $a \geq 0$ and $k > 1$. The case $a = 0$ and k odd is analyzed in detail, giving a complete characterization of its structure and behavior, marking the first time that such a parametric family of meta-Fibonacci sequences has been solved. This behavior is considerably more complex than that of the more familiar Conolly sequence ($a = 0$, $k = 2$). Various properties are derived: for example, a certain difference of summands turns out to consist of palindromic subsequences, and the mean values of the functions on these subsequences are computed. Conjectures are made concerning the still more complex behavior of $a = 0$ and even $k > 2$.

Key words. Hofstadter, iterated recursion, meta-Fibonacci, Q sequence

AMS subject classifications. 11B37, 11B39, 11B99

DOI. 10.1137/S0895480103421397

1. Introduction.

In this paper, all values are integers and defining equations use the ":=" operator. For a sequence T and an integer d, we write $\Delta_d T(n) := T(n) - T(n-d)$.

Hofstadter's Q sequence $Q(n)$, illustrated in Figure 1.1, first mentioned in [7] and defined by the "self-referencing" recursion

$$Q(n) := Q(n - Q(n-1)) + Q(n - Q(n-2)), \quad n > 2, \tag{1.1}$$

and initial conditions $Q(1) := Q(2) := 1$ is the most famous example of a so-called *meta-Fibonacci sequence*. This sequence, recently renamed $U(n)$ by Hofstadter and his collaborators [9], remains the focus of ongoing investigation in [8] and elsewhere, although to date very little has been proven about its enigmatic behavior.

At the same time, various authors have examined seemingly close relatives to the above recursion, which have turned out to be far better behaved and about which a great deal can be demonstrated. In [3], Conolly introduced the following very well-behaved variant of the Q-sequence recursion, illustrated in Figure 1.2:

$$F(n) := F(n - F(n-1)) + F(n - 1 - F(n-2)), \quad n > 2, \tag{1.2}$$

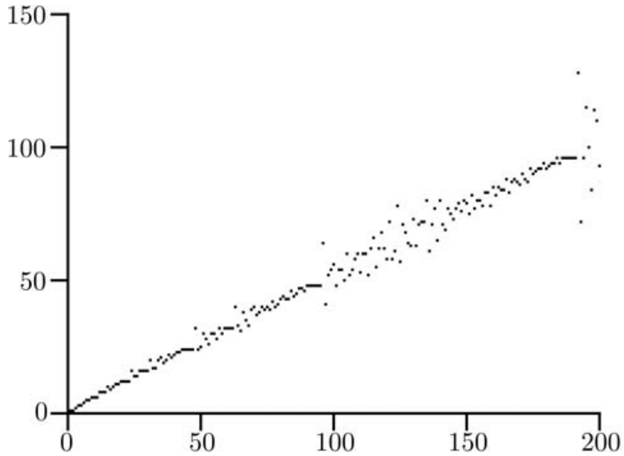
with initial conditions $\{F(1) = 0,\ F(2) = 1\}$ or $\{F(1) = 1,\ F(2) = 1\}$. He notes that for $n > 2$ the recursion yields the same sequence whether $F(1) = 0$ or $F(1) = 1$.

Much can be said about this sequence. For example, Conolly shows that if $F(1) = 0$ and $n = 2^i + j$ with $i \geq 1$ and $0 \leq j < 2^i$, then

$$F(n) = 2^{i-1} + F(j+1). \tag{1.3}$$

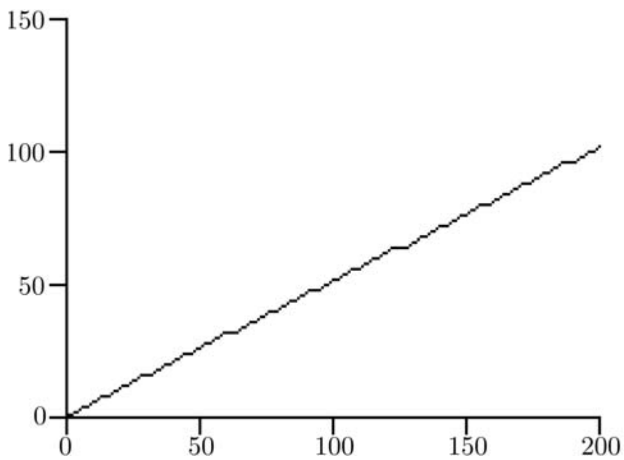
*Received by the editors January 16, 2003; accepted for publication (in revised form) November 3, 2004; published electronically May 13, 2005.
http://www.siam.org/journals/sidma/18-4/42139.html

†Department of Mathematics, University of Toronto, Toronto, ON, M5S 3G3, Canada (calaghan@math.utoronto.ca, jjchew@math.utoronto.ca, tanny@math.utoronto.ca).



n	$Q(n)$	n	$Q(n)$
1	1	13	8
2	1	14	8
3	2	15	10
4	3	16	9
5	3	17	10
6	4	18	11
7	5	19	11
8	5	20	12
9	6	21	12
10	6	22	12
11	6	23	12
12	8	24	16

FIG. 1.1. *Hofstadter's Q sequence.*



n	$F(n)$	n	$F(n)$
1	1	13	8
2	1	14	8
3	2	15	8
4	2	16	8
5	3	17	9
6	4	18	10
7	4	19	10
8	4	20	11
9	5	21	12
10	6	22	12
11	6	23	12
12	7	24	13

FIG. 1.2. *Conolly's F sequence.*

Tanny [14] and Higham and Tanny [5, 6] developed a considerably more extensive analysis of the very similar recursion, illustrated in Figure 1.3, defined by

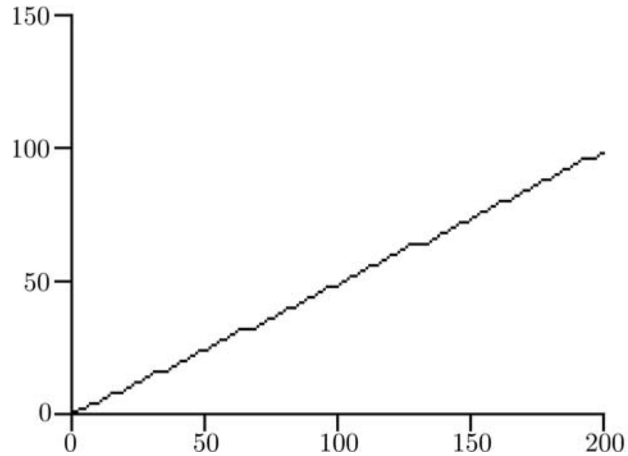
$$T(n) := T(n-1-T(n-1)) + T(n-2-T(n-2)), \quad n > 2, \tag{1.4}$$

with initial conditions $T(0) := T(1) := T(2) := 1$, which generates a sequence with highly analogous properties. Detailed results were provided concerning the structure and behavior of this sequence, and of some other sequences generated by (1.4) but with alternative choices for the initial conditions.

In [5], a k-term generalization of (1.4) was suggested:

$$T_k(n) := \sum_{i=1}^{k} T_k(n-i-T_k(n-i)), \quad n > k. \tag{1.5}$$

It was proven there that with the initial conditions $T_k(0) := T_k(1) := 1$ and $T_k(i) := i-1,\ 2 \le i \le k$, the sequence generated by (1.5) behaves in very much the



n	$T(n)$	n	$T(n)$
1	1	13	6
2	1	14	7
3	2	15	8
4	2	16	8
5	2	17	8
6	3	18	8
7	4	19	8
8	4	20	9
9	4	21	10
10	4	22	10
11	5	23	11
12	6	24	12

FIG. 1.3. *Tanny's T sequence.*

same way as the sequence generated by (1.4) with initial conditions $T(0) := T(1) := T(2) := 1$. That is, it is monotone, its consecutive terms increase by either 0 or 1, and it hits every positive integer. Further, we can determine explicit formulae that characterize its behavior.

The work cited above with (1.5) demonstrates that recursions expressible as homogeneous sums may be approachable even when the sums have many terms. Motivated by this success, we introduce an even more general formulation of the Conolly recursion (1.2) that incorporates all of the specific variants discussed above.

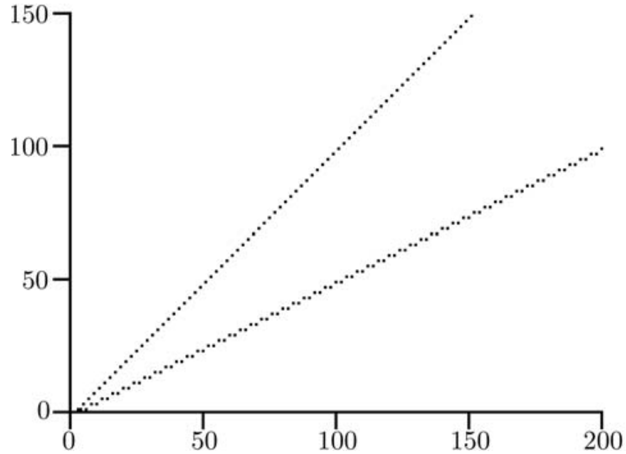
For $a \geq 0$ and $k \geq 2$ consider the family of recursions

$$T_{a,k}(n) := \sum_{i=0}^{k-1} T_{a,k}(n - i - a - T_{a,k}(n - i - 1)), \quad n > a + k. \tag{1.6}$$

Equation (1.6) reduces to (1.2) when $a = 0$ and $k = 2$, to (1.4) when $a = 1$ and $k = 2$, and to (1.5) when $a = 1$. For $k = 2$ and arbitrary (even negative) a, Elzinga [4] reports that some analogues of certain results of (1.5) appear to hold, although no proof is provided; again, for $k = 2$ and $a \in \{-1, -2\}$, he describes the behavior of (1.6) for a substantial number of sets of initial conditions in an attempt to identify a behavioral classification scheme. For $k = 3$ and $a = 0$, Allenby and Smith [1] present some initial results and conjectures.

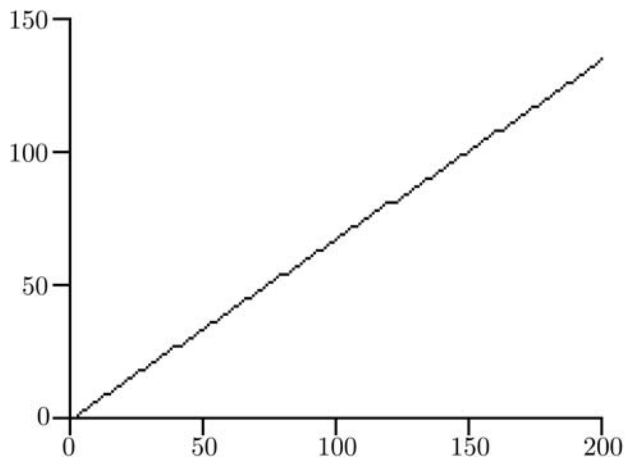
As is typical for meta-Fibonacci recursions, the behavior of the individual members of this family is highly sensitive to the choice of the parameters a and k and to the initial conditions. Some choices lead to sequences with identifiable and regular (though potentially very complex) patterns, while others generate highly chaotic sequences or even cause the sequence $T_{a,k}(n)$ to fail to be defined for some n.

For example, consider the choice of initial values for the sequence $T_{0,3}$. If we require that each of the three summands that make up the recursive definition (1.6) of $T_{0,3}(4)$ evaluates to one of the chosen initial values, then the three conditions $1 \leq 4 - i - T_{0,3}(3 - i) \leq 3$, $i \in \{1, 2, 3\}$, allow 27 possible sets of initial values ranging lexicographically from $(T_{0,3}(1), T_{0,3}(2), T_{0,3}(3)) = (-1, 0, 1)$ to $(1, 2, 3)$. Four of the 27 choices—$(-1, 0, 2)$, $(-1, 0, 3)$, $(-1, 2, 3)$, and $(1, 0, 3)$—give a $T_{0,3}(n)$ which is not well defined for some $5 \leq n \leq 7$. In the other 23 cases, we have verified empirically that $T_{0,3}(n)$ is defined at least up to $n = 10{,}000$.



n	$T(n)$	n	$T(n)$
1	0	13	12
2	0	14	6
3	2	15	14
4	2	16	8
5	4	17	16
6	2	18	8
7	6	19	18
8	4	20	10
9	8	21	20
10	4	22	10
11	10	23	22
12	6	24	12

FIG. 1.4. $T_{0,3}$ *with initial values* $(0, 0, 2)$.

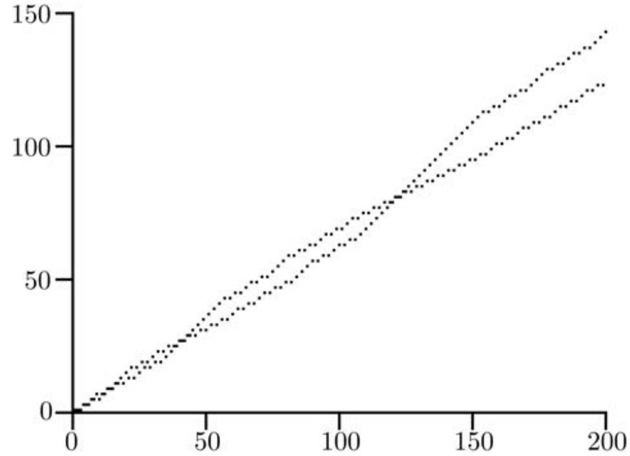


n	$T(n)$	n	$T(n)$
1	0	13	9
2	0	14	9
3	1	15	9
4	2	16	10
5	3	17	11
6	3	18	12
7	4	19	12
8	5	20	13
9	6	21	14
10	6	22	15
11	7	23	15
12	8	24	16

FIG. 1.5. $T_{0,3}$ *with initial values* $(0, 0, 1)$.

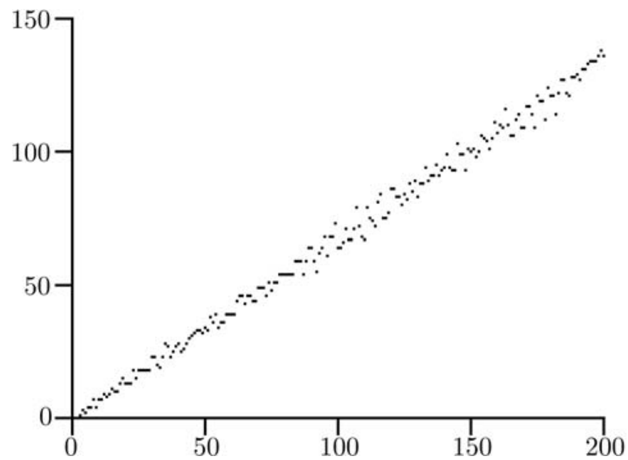
Four of the remaining 23 cases—$(0, 0, 2)$, $(0, 0, 1)$, $(1, 1, 1)$, and $(-1, 0, 1)$—are graphed and tabulated in Figures 1.4, 1.5, 1.6, and 1.7 and illustrate the wide range of possible behavior for these sequences. Observe that the first and third sequences visibly bifurcate into an even subsequence and an odd subsequence, each of which is monotonic even though the full sequence is not. The second sequence appears to be very similar to Conolly's sequence, while the fourth sequence looks completely chaotic.

It is also typical of meta-Fibonacci recursions that alternative choices for the parameters and initial conditions can lead to essentially the same sequence. In this vein, it is not difficult to show that if $T(n)$ is always well defined and positive, and some other $T^*(n)$ defined according to (1.6) satisfies $T^*(N + p) = T(p)$ for an $N \geq 0$ and $1 \leq p \leq a + k$, then $T^*(N + p) = T(p)$ for all $p \geq 1$. For instance, setting $a = 0$, $k = 3$, and $\{T(n)\}_{n=1,2,3} = (-1, 1, 2)$ or $(1, 1, 1)$ gives two sequences that are identical except that one has four extra values at its start. We also note that we may replace n by $n' + x$ in (1.6) to shift a sequence x places.



n	$T(n)$	n	$T(n)$
1	1	13	9
2	1	14	9
3	1	15	9
4	3	16	11
5	3	17	11
6	3	18	13
7	5	19	11
8	5	20	15
9	7	21	13
10	5	22	17
11	7	23	13
12	7	24	17

FIG. 1.6. $T_{0,3}$ *with initial values* $(1, 1, 1)$.



n	$T(n)$	n	$T(n)$
1	-1	13	8
2	0	14	9
3	1	15	11
4	3	16	10
5	2	17	10
6	4	18	13
7	4	19	15
8	7	20	13
9	4	21	13
10	7	22	13
11	7	23	18
12	9	24	15

FIG. 1.7. $T_{0,3}$ *with initial values* $(-1, 0, 1)$.

In this paper, we analyze the general family defined by (1.6) and initial conditions $T_{a,k}(n) = 1$ for $1 \leq n \leq a + k$. We choose this set of initial values because, as shown in Figure 1.6, its bifurcated, undulating behavior is complex enough to be interesting while still structured enough to be amenable to rigorous analysis.

In section 2 we prove that for general a, the sequence generated by each recursion in the family can be decomposed into k disjoint subsequences, each of which begins with 1 and increases monotonically in increments of 0 or $k - 1$. Further, for all odd k, the differences $T_{a,k}(n) - T_{a,k}(n-2)$ are also always either 0 or $k - 1$. This finding substantially generalizes an earlier result of this type for (1.4) appearing in Higham and Tanny [5, 6].

In sections 3 and 4, we apply and extend the results of section 2 to derive a detailed characterization of the behavior of (1.6) in the special case $a = 0$, $k = 3$ and explore the many interesting properties and beautiful symmetries that it exhibits. This case illustrates well the additional complexity that occurs once k is increased beyond 2, yet remains manageable for expository purposes. In the course of our work,

TABLE 2.1
$T_{0,7}(n)$.

	n							n					
	1	2	3	4	5	6		1	2	3	4	5	6
$T(n+0)$	1	1	1	1	1	1	$T(n+60)$	49	49	49	55	55	55
$T(n+6)$	1	7	7	7	7	7	$T(n+66)$	55	55	55	61	55	61
$T(n+12)$	7	7	13	13	13	13	$T(n+72)$	61	61	61	67	61	67
$T(n+18)$	13	13	19	13	19	19	$T(n+78)$	61	67	67	73	67	73
$T(n+24)$	19	19	25	19	25	19	$T(n+84)$	67	73	67	79	73	79
$T(n+30)$	25	25	31	25	31	25	$T(n+90)$	73	79	73	85	73	85
$T(n+36)$	31	25	31	31	37	31	$T(n+96)$	79	85	79	91	79	91
$T(n+42)$	37	31	37	37	37	37	$T(n+102)$	79	91	85	97	85	97
$T(n+48)$	43	37	43	43	43	43	$T(n+108)$	85	97	85	103	91	103
$T(n+54)$	43	43	49	49	49	49	$T(n+114)$	91	103	91	109	91	109

we considerably extend the results on this case that appear in [1] and settle all the conjectures stated there.

While the case $a = 0$, $k = 3$ is interesting in its own right, what is more exciting is that the behavior in this case appears to provide a roadmap for understanding and characterizing the detailed behavior of the sequences for all odd k. This is the content of section 5, where we prove that certain key results derived for $a = 0$, $k = 3$ hold for all odd k. On this basis, we show that the detailed structure for all odd k is analogous to that shown for $k = 3$. To the best of our knowledge, this is the first instance in the area of meta-Fibonacci recursions where such broadly based behavioral results have been shown to hold for sequences with such complex behavior.

For even $k \geq 4$ the behavior of $T(n)$ is much more complicated and erratic than for odd k (the case $k = 2$, the Conolly sequence discussed earlier, is completely understood and quite straightforward). There is evidence that the same kind of approaches that we used for odd k can be adapted for even k, although we have not yet investigated this very far. In section 6, we conclude with some conjectures relating to even k.

2. A family of meta-Fibonacci sequences. In this section, we prove some results concerning the general recursion

$$T_{a,k}(n) := \sum_{i=0}^{k-1} T_{a,k}\big(n - i - a - T_{a,k}(n-i-1)\big), \quad n > a+k, \tag{1.6}$$

$$T_{a,k}(n) := 1, \quad 1 \leq n \leq k+a, \tag{2.1}$$

which for brevity's sake we will refer to simply as $T(n)$.

Table 2.1 lists some values of $T(n)$ for the case $a = 0$, $k = 7$ and lets us observe a few of its properties: $T(n) \equiv 1 \pmod{k-1}$; $T(n)$ is not monotonic but *is* composed of $k-1$ interleaved monotonic subsequences $\{T((k-1)i+j)\}_{i=0}^{\infty}$, $j = 1, \ldots, k-1$; and each of these subsequences includes every possible value subject to the modulo constraint. That is, $\Delta_{k-1}T(n) := T(n) - T(n-k+1) \in \{0, k-1\}$ for all $n \geq k$. Furthermore, in the case of odd k, we will find that $\Delta_d T(n) := T(n) - T(n-d) \in \{0, k-1\}$ for all even $d \in [0, k-1]$.

We begin our analysis of $T(n)$. In the case of an ordinary Fibonacci sequence, the recursive definition of a sequence member refers simply to immediately preceding sequence members. In the meta-Fibonacci sequence $T(n)$, the recursive summands are usually much earlier sequence members, whose distance from the current n can vary

TABLE 2.2
$U_{0,7}(n)$.

	n							n					
	1	2	3	4	5	6		1	2	3	4	5	6
$U(n+0)$	–	1	1	1	1	1	$U(n+60)$	7	7	7	13	7	7
$U(n+6)$	1	1	1	1	1	1	$U(n+66)$	7	7	7	13	7	13
$U(n+12)$	1	1	7	1	1	1	$U(n+72)$	7	7	7	13	7	13
$U(n+18)$	1	1	7	1	7	1	$U(n+78)$	7	13	7	13	7	13
$U(n+24)$	1	1	7	1	7	1	$U(n+84)$	7	13	7	19	7	13
$U(n+30)$	7	1	7	1	7	1	$U(n+90)$	7	13	7	19	7	19
$U(n+36)$	7	1	7	7	7	1	$U(n+96)$	7	13	7	19	7	19
$U(n+42)$	7	1	7	7	7	7	$U(n+102)$	7	19	7	19	7	19
$U(n+48)$	7	1	7	7	7	7	$U(n+108)$	7	19	7	25	7	19
$U(n+54)$	7	7	7	7	7	7	$U(n+114)$	7	19	7	25	7	25

considerably within the sum. We therefore need to examine these summands more closely and reword the definition (1.6) of $T_{a,k}(n)$ to label two functions of subsequent interest:

$$T_{a,k}(n) = \sum_{i=0}^{k-1} U_{a,k}(n-i), \quad n > a+k, \tag{2.2}$$

where

$$U_{a,k}(n) := T_{a,k}(R_{a,k}(n)), \quad n > a+1,$$
$$R_{a,k}(n) := n - a - T_{a,k}(n-1), \quad n > 1.$$

As with $T(n)$, we abbreviate $U_{a,k}(n)$ to $U(n)$ and $R_{a,k}(n)$ to $R(n)$ wherever the understood subscripts are clear. Returning to the case $a = 0$, $k = 7$, Table 2.2 lists the first 119 values of $U(n)$ and shows that it has the same modulo and subsequence properties that $T(n)$ has; that is, $\Delta_{k-1}U(n)$ takes on only the two values 0 and $k-1$.[1] Table 2.3 lists the first 119 values of $R(n)$, which also shares all of these properties with $T(n)$ and $U(n)$.

Note 2.1. We can make a stronger observation for $\Delta_6 U(n)$ that turns out to generalize to any k: every pair of 6's in this difference sequence appears to be separated by at least seven zeros. The following are the first 88 values of $\Delta_6 U(n)$, listed with a break before each 6: $\{\Delta_6 U(n)\}_{n=8}^{95} = \{$0,0,0,0,0,0,0, 6,0,0,0,0,0,0,0, 6,0,0,0,0,0,0,0, 6, 0,0,0,0,0,0,0,0, 6,0,0,0,0,0,0,0, 6,0,0,0,0,0,0,0, 6,0,0,0,0,0,0,0, 6,0,0,0,0,0,0,0, 6,0,0,0,0, 0,0,0, 6,0,0,0,0,0,0,0, 6,0,0,0,0,0,0,0$\}$.

Since $\Delta_{k-1}T(n) = \sum_{i=0}^{k-1} \Delta_{k-1}U(n-i)$, it follows that if we can fully describe $\Delta_{k-1}U(n)$, then we have described its double sum $T(n)$. In fact we will demonstrate that, as is illustrated in Note 2.1, $\Delta_{k-1}U(n)$ has a beautiful, highly regular structure for general k from which many properties of $\Delta_{k-1}T(n)$ and hence $T(n)$ can readily be deduced. For example, if Note 2.1 is generally true for $k = 7$, then $\Delta_6 T(n)$ also takes on only the values 0 and 6. We may thus conclude that (unlike many meta-Fibonacci sequences) $T(n)$ is well defined for all n.

We now show that this is true for general k.

PROPOSITION 2.2. *The following differences are all either* 0 *or* $k-1$*:* $\Delta_{k-1}T(n)$ *for* $n \geq k$*,* $\Delta_{k-1}R(n)$ *for* $n > k$*, and* $\Delta_{k-1}U(n)$ *for* $n > k+a$*. As a result, the*

[1] $\Delta_2 U_{0,3}(n)$ is what Allenby and Smith [1] call a pairing, though they neither explicitly define the sequence $\{\Delta_2 U_{0,3}(n)\}$ nor make it the focus of their analysis as we do here.

Table 2.3
$R_{0,7}(n)$.

	n							n					
	1	2	3	4	5	6		1	2	3	4	5	6
$R(n+0)$	-	1	2	3	4	5	$R(n+60)$	12	13	14	15	10	11
$R(n+6)$	6	7	2	3	4	5	$R(n+66)$	12	13	14	15	10	17
$R(n+12)$	6	7	8	3	4	5	$R(n+72)$	12	13	14	15	10	17
$R(n+18)$	6	7	8	3	10	5	$R(n+78)$	12	19	14	15	10	17
$R(n+24)$	6	7	8	3	10	5	$R(n+84)$	12	19	14	21	10	17
$R(n+30)$	12	7	8	3	10	5	$R(n+90)$	12	19	14	21	10	23
$R(n+36)$	12	7	14	9	10	5	$R(n+96)$	12	19	14	21	10	23
$R(n+42)$	12	7	14	9	10	11	$R(n+102)$	12	25	14	21	10	23
$R(n+48)$	12	7	14	9	10	11	$R(n+108)$	12	25	14	27	10	23
$R(n+54)$	12	13	14	9	10	11	$R(n+114)$	12	25	14	27	10	29

following sequence values are all (well) defined: $T(n)$ for $n > 0$, $R(n)$ for $n > 1$, and $U(n)$ for $n > a+1$.

Proof. We begin an inductive proof by verifying that the proposition holds for $n \le 2k + a$.

$T(n) = 1$ for $1 \le n \le k + a$ by (2.1). It is easy to compute using (1.6) and (2.1) that $T(n) = k$ for $k + a < n \le 2k + a$. $\Delta_{k-1}T(n)$ is thus either 0 or $k-1$ for $k \le n \le 2k + a$.

$R(n) := n - a - T(n-1)$ is $n - a - 1$ for $2 \le n \le k + a + 1$ and is $n - a - k > 0$ for $k+a+1 < n \le 2k+a+1$. $\Delta_{k-1}R(n) = k-1-\Delta_{k-1}T(n-1)$ is either 0 or $k-1$ for $k+1 \le n \le 2k+a+1$.

$U(n) := T(R(n))$. When $k + a + 1 \le n \le 2k + a$, we have $1 \le R(n) \le k$. When $1 \le k \le n$, we have $T(n) = 1$. So when $k + a + 1 \le n \le 2k + a$, we have $U(n) = T(R(n)) = 1$. Thus, $\Delta_{k-1}U(n) = 0$ is for $k + a + 1 \le n \le 2k + a$.

Now let $n > 2k + a$ and proceed assuming that the proposition holds for all lesser n.

$R(n) := n - a - T(n-1)$ is defined.

$\Delta_{k-1}R(n) = k - 1 - \Delta_{k-1}T(n-1)$ is either $k-1$ or 0, since $\Delta_{k-1}T(n-1)$ is either 0 or $k-1$.

$U(n) := T(R(n))$. To confirm that $U(n)$ is well defined, we need to show that $R(n)$ is well defined (which we just did) and that T is well defined at $R(n)$. That is, we require that $0 < R(n) < n$. Let $m \in [k+a+2, 2k+a+1]$ with $m \equiv n \pmod{k-1}$. Then $R(n) = R(m)+\Delta_{k-1}R\big(m+k-1\big)+\Delta_{k-1}R\big(m+2(k-1)\big)+\cdots+\Delta_{k-1}R\big(n\big)$. The first summand is positive and the rest are all either 0 or $k-1$, so $R(n)$ is positive. Likewise, $T(n) > 0$, so $R(n) := n - a - T(n-1) < n$. Therefore, we can apply induction to find that $U(n)$ is defined.

$\Delta_{k-1}U(n) = T(R(n)) - T(R(n-k+1))$. We just showed that $R(n)-R(n-k+1) = \Delta_{k-1}R(n)$ is either 0 or $k-1$. If this difference is 0, then so is $\Delta_{k-1}U(n)$. If it is $k-1$, then $\Delta_{k-1}U(n) = (\Delta_{k-1}T)(R(n))$, which is 0 or $k-1$ by the induction assumption.

From (2.2), we have $T(n) = T(n-1) + U(n) - U(n-k)$, so $T(n)$ is well defined.

$\Delta_{k-1}T(n) = \sum_{i=0}^{k-1} \Delta_{k-1}U(n-i)$ from (2.2) and the linearity of the difference operator. If all the summands are zero, then so is $\Delta_{k-1}T(n)$ and we are done. If not, let j be the largest integer in $[n-k+1, n]$ for which $k-1 = \Delta_{k-1}U(j) = T(R(j)) - T(R(j-k+1))$. Then $\Delta_{k-1}R(j) = k-1$ and $0 = k-1-\Delta_{k-1}R(j) = \Delta_{k-1}T(j-1) = \sum_{i=0}^{k-1} \Delta_{k-1}U(j-1-i)$. Therefore at most one of $\Delta_{k-1}U(n-k+1)$, $\ldots$, $\Delta_{k-1}U(n)$ can have the value $k-1$ while the others must be 0, so their sum $\Delta_{k-1}T(n)$ is either 0 or $k-1$. $\square$

Note that other choices of initial values for T will prevent the initial conditions of the induction from being satisfied and that in such cases $\Delta_{k-1}U(n)$ and $\Delta_{k-1}T(n)$ may belong to a larger finite set (see Figure 1.4, or the case $a = 0$, $k = 3$ with initial values $\{0, 2, 1\}$) or take on an infinite number of different values (see Figure 1.7).

COROLLARY 2.3. *For $n > 2k+a+1$, if $\Delta_{k-1}T(n) = 0$, then $\Delta_{k-1}U(n-k+1) = \Delta_{k-1}U(n-k+2) = \cdots = \Delta_{k-1}U(n) = 0$.*

Proof. If $0 = \Delta_{k-1}T(n) = \sum_{j=0}^{k-1} \Delta_{k-1}U(n-j)$ and each summand is either 0 or $k-1$, each summand must be zero. □

COROLLARY 2.4. *For each j in $[1, k]$, the subsequence $\{T((k-1)i+j)\}_{i=0}^{\infty}$ begins with 1 and increases monotonically in increments of 0 or $k-1$.*

Proof. The result follows immediately from the initial conditions for T and what we have shown about the differences $\Delta_{k-1}T(n)$. □

COROLLARY 2.5. *In the sequence $\Delta_{k-1}U(n)$, $n > a+k$, there are at least k zeros between each pair of nonzero values.*

Proof. We showed in the proof of Proposition 2.2 that $\Delta_{k-1}U(n) = 0$ for $k+a < n \leq 2k+a$ and if for some $j > 2k+a$ the jth member of $\Delta_{k-1}U$ is nonzero, then it is preceded by at least k zeros. □

The preceding corollary describes the fine structure of $\Delta_{k-1}U(n)$ and helps explain why that sequence attracts our interest. We conclude this section by formally defining the blocks that make up the structure of the $\Delta_{k-1}U(n)$ sequence.

DEFINITION 2.6. *A* p-block *is a consecutive subsequence of length p in $\{\Delta_{k-1}U(n)\}$, consisting of an initial nonzero value together with all of the zeros which follow it.*

Example 2.7. The values of $\Delta_6U(n)$ listed in Note 2.1 constitute seven zeros, followed by two 8-blocks, one 9-block, and seven more 8-blocks.

3. The structure of a three-term recursion. We move now from general observations about properties of our family of difference functions to the specific case $k = 3$; observe that this is a three-term generalization of Conolly's sequence (1.2). This case illustrates the increased structural complexity that occurs for $k > 2$. It also demonstrates concretely the behavior typical of odd k, to be elaborated in section 5.[2]

In (1.6), set $a = 0$, $k = 3$ and assume the initial conditions $T_{0,3}(n) = 1$ for $n = 1, 2, 3$. For the remainder of this section and in the next section, we refer to this sequence as $T(n)$, to $R_{0,3}(n)$ as $R(n)$, and to $U_{0,3}(n)$ as $U(n)$:

$$
\begin{aligned}
&T(n) := U(n) + U(n-1) + U(n-2), \quad n > 3,\\
&U(n) := T(R(n)), \quad n > 1, \qquad\qquad (3.1)\\
&R(n) := n - T(n-1), \quad n > 1.
\end{aligned}
$$

Table 3.1 lists values of $T(n)$, $U(n)$, and $R(n)$ for small n.

Note 3.1. For future reference, we list the values of $\{\Delta_2U(n)\}_{n=4}^{97} = \{$0,0,0, 2,0,0,0,0, 2,0,0,0, 2,0,0,0, 2,0,0,0,0, 2,0,0,0,0, 2,0,0,0,0, 2,0,0,0, 2,0,0,0, 2,0,0,0, 2,0,0,0, 2,0,0,0, 2,0,0,0,0, 2,0,0,0,0, 2,0,0,0,0, 2,0,0,0,0, 2,0,0,0, 2,0,0,0,0, 2,0,0,0, 2,0,0,0,0, 2,0,0,0,0$\}$.

We completely describe the behavior of T, as well as the related sequences U and R. As is the case with many other meta-Fibonacci sequences, the T sequence can be naturally decomposed into strings of consecutive terms called *generations*. The generational structure is clearly visible in Figure 1.6, where the end of each

[2]Using very different methods, Allenby and Smith [1] analyze this recursion together with several other, less complicated, meta-Fibonacci recursion variants. In this and the next section, we extend considerably the results proven in [1] and settle all the conjectures stated therein.

TABLE 3.1
$T(n)$, $U(n)$, $R(n)$ for $a = 0$, $k = 3$.

n	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39
$T(n)$	1	1	3	5	7	7	9	9	11	11	13	13	15	17	17	19	19	21	23	25
$T(n+1)$	1	3	3	5	5	7	9	11	13	15	17	17	19	19	21	23	23	25	25	27
$U(n)$	–	1	1	3	3	3	3	3	3	3	3	3	5	5	5	5	5	7	7	9
$U(n+1)$	1	1	1	1	1	3	3	5	5	7	7	7	7	7	9	9	9	9	9	9
$R(n)$	–	2	2	4	4	6	6	6	6	6	6	6	8	8	10	10	10	12	12	14
$R(n+1)$	1	3	3	3	3	5	5	7	7	9	9	11	11	11	13	13	15	15	15	15

generation is marked by several points of intersection of the two curves. The lengths of the generations increase geometrically, and properties of the sequence recur from generation to generation giving a structure which might be described as *logarithmic-periodic*.

To describe completely the behavior of $T(n)$ it is sufficient to characterize fully the behavior of $\Delta_2 U(n)$, since $T(n)$ can be reconstructed as a double sum of $\Delta_2 U(n)$. Indeed, the generational structure of $T(n)$ is mirrored in a surprising way in a corresponding generational structure in the $\Delta_2 U(n)$ sequence. Even further, we will show that each of these corresponding generations in $\Delta_2 U$ is in turn made up of substrings we call *lines* and subsubstrings we call *feet*. That is, we will show how $\Delta_2 U(n)$ consists of feet, concatenations of which form lines, and how consecutive lines complete the generation. We also show that the feet in each generation in $\Delta_2 U(n)$ are determined by (and in fact are in the one-to-one correspondence shown in Figure 3.1 with) the values of $\Delta_2 T$ in the immediately preceding generation. In this way, the preceding generation of T determines new feet, which make up new lines; these new lines make up a new generation in $\Delta_2 U$, which further determines the new generation of T.

From Note 3.1, we empirically observe (and will show below in Proposition 3.3) that from $n = 7$ onward $\Delta_2 U(n)$ appears to consist only of 4-blocks and 5-blocks. We will see that the interesting values of n are those which start a 4-block, start a 5-block, or end a 5-block, and that it is logical then to look at $\Delta_2 U$ as consisting mostly of subsequences $\{2, 0, 0, 0\}$ with the occasional extra $\{0\}$.

Perhaps not surprisingly, the ancient Greeks [2, 11] had words for such patterns (feet) of one stressed beat followed by three unstressed beats (a first paeon), for a sequence of such patterns (a line), and for an extra unstressed beat at the end of a sequence (a hypercatalectic). While Greek prosodists were concerned with long and short syllables in poetry and we are dealing with 2's and 0's in an iterated recursion, the analogy is precise, and so we borrow their terminology to define our first type of subsequence.

DEFINITION 3.2 (foot). *A paeon is a sequence* $\{2,0,0,0\}$ *of consecutive values of* $\Delta_2 U(n)$. *A hypercatalectic is a singleton sequence* $\{0\}$*, immediately preceded in* $\{\Delta_2 U(n)\}$ *by a paeon. A foot is either a paeon or a hypercatalectic. For convenience, we will write* $\{\mathrm{P}\}$ *interchangeably with* $\{2, 0, 0, 0\}$ *and likewise* $\{\mathrm{H}\}$ *with* $\{0\}$ *when listing values of* $\Delta_2 U$. *We also define* $\varphi(n)$ *as the symbol* P *if* $\Delta_2 U(n) = 2$ *begins a paeon and* H *if* $\Delta_2 U(n) = 0$ *is a hypercatalectic, and we leave it undefined otherwise.*

So $\varphi(7) = \varphi(12) = \varphi(16) = \varphi(20) = P$, $\varphi(11) = \varphi(24) = H$ and $\varphi(n)$ is not defined for other $7 \leq n \leq 24$.

PROPOSITION 3.3. $\{\Delta_2 U(n)\}_{n=7}^{\infty}$ *consists only of feet.*

Proof. By Definition 3.2, this is equivalent to saying that there are no p-blocks of length $p > 5$ in $\Delta_2 U(n)$ or that if $\Delta_2 U(n) = 2$ and $\Delta_2 U(n + 4) = 0$, then $\Delta_2 U(n + 5) = 2$.

We proceed to prove the latter by induction. The enumeration in Note 3.1 shows that the proposition holds for $n = 7$. In particular, $\Delta_2 U(7) = \Delta_2 U(12) = 2$ and $\Delta_2 U(11) = 0$. Assume that the proposition is true for all n less than some $N > 7$, that $\Delta_2 U(N) = 2$, and that $\Delta_2 U(N+4) = 0$. Corollary 2.5 and induction tell us that for some nonnegative q, $\Delta_2 U(N) = \Delta_2 U(N-4) = \cdots = \Delta_2 U(N-4q) = \Delta_2 U(N-4q-5) = 2$ and $\Delta_2 U(i) = 0$ for all other i in the interval $[N-4q-5, N+4]$.

We can compute differences of T by expanding into these known values of $\Delta_2 U$: $T(N+4) - T(N+2) = \Delta_2 T(N+4) = \sum_{i=N+2}^{N+4} \Delta_2 U(i) = 0$, $T(N-4q-2) - T(N-4q-4) = \Delta_2 T(N-4q-2) = \sum_{i=N-4q-4}^{N-4q-2} \Delta_2 U(i) = 0$, and $T(N+2) - T(N-4q-2) = \sum_{i=1}^{2q+2} \Delta_2 T(N-2i+4) = \sum_{i=1}^{2q+2} \sum_{j=0}^{2} \Delta_2 U(N-2i-j+4) = 4q+4$.

So $T(N+4) = T(N-4q-4) + 4q + 4$, $R(N+5) = N + 5 - T(N+4) = 2 + N + 3 - T(N+2) = R(N+3) + 2$, $R(N+5) = N - 4q + 1 - T(N-4q-4) < N$, and $R(N+5) = 2 + N - 4q - 1 - T(N-4q-2) = R(N-4q-1) + 2$.

Let $y := R(N+5)$. Then $\Delta_2 T(y-2) = \Delta_2 U(N-4q-1) = 0$. By Corollary 2.3, $\Delta_2 U(y-2) = \Delta_2 U(y-3) = \Delta_2 U(y-4) = 0$. By induction, $\Delta_2 U(y-1)$ and $\Delta_2 U(y)$ cannot both be zero, or else $\Delta_2 U$ would have five consecutive zeros somewhere before N, at $y-4$, $y-3$, $y-2$, $y-1$, and y. Therefore, $\Delta_2 U(y) + \Delta_2 U(y-1) + \Delta_2 U(y-2) = 2 = \Delta_2 T(y) = U(N+5) - T(R(N+5)-2) = U(N+5) - U(N+3) = \Delta_2 U(N+5)$ as required. ∎

Note that as with Proposition 2.2, other choices of initial values for T (such as $\{T(n)\}_{n=1}^{3} := \{1, 2, 1\}$) will render false the initial conditions for the induction in the proof of Proposition 3.3. In such cases, the spacing between nonzero members of $\{\Delta_2 U(n)\}$ can grow without bound.

We now use our results so far to create Table 3.2, a tabulation of six of our functions over the course of a paeon beginning at n_0, followed possibly by a hypercatalectic. We express the values of the functions in terms of the boxed parameters t_0, t_1, r_0, r_1, and d_4. (The paeon is followed by a hypercatalectic iff $d_4 = 0$, that is, $\varphi(n_0+4) = H$ iff $d_4 = 0$.)

This description of the intricate local relationship among the T, R, and U sequences tells us how to easily compute the parameter values in one foot of $\Delta_2 U$ from the parameter values in the preceding foot. Further, it will be fundamental to an understanding of how the occurrence of feet in $\Delta_2 U$ relates to much earlier values of

Table 3.2
Function values over a paeon.

n	$T(n)$	$\Delta_2 T(n)$	$R(n)$	$\Delta_2 R(n)$	$\Delta_2 U(n)$	$\varphi(n)$
n_0-3	t_1-2	2	?	0	0	–
n_0-2	t_0-2	?	r_0-2	0	0	–
n_0-1	t_1-2	0	r_1	?	0	?
$\boxed{n_0}$	$\boxed{t_0}$	2	$\boxed{r_0}$	2	2	P
n_0+1	$\boxed{t_1}$	2	$\boxed{r_1}$	0	0	–
n_0+2	t_0+2	2	r_0	0	0	–
n_0+3	t_1	0	r_1	0	0	–
n_0+4	t_0+d_4+2	d_4	r_0+2	2	$\boxed{d_4}$	P or H
n_0+5	t_1+2	2	r_1+2-d_4	$2-d_4$	$2-d_4$	– or P
n_0+6	t_0+d_4+4	2	r_0+2	0	0	–
n_0+7	t_1-d_4+4	$2-d_4$	r_1+2-d_4	0	0	–

$\Delta_2 T$. This then constitutes the first of our three structural theorems.

THEOREM 3.4 (foot pattern theorem). *Suppose the parameters* n_0, t_0, t_1, r_0, r_1, *and* d_4 *satisfy all of the following conditions:* $n_0 \geq 7$, $T(n_0) = t_0$, $T(n_0+1) = t_1$, $R(n_0) = r_0$, $R(n_0+1) = r_1$, $\Delta_2 U(n_0) = 2$, *and* $\Delta_2 U(n_0+4) = d_4$. *Then* $T(n)$, $\Delta_2 T(n)$, $R(n)$, $\Delta_2 R(n)$, $\Delta_2 U(n)$, *and* $\varphi(n)$ *have the values shown in Table* 3.2.

Proof. Use Proposition 3.3 and Corollary 2.5 to calculate $\Delta_2 U(n)$. Apply the Δ_2 operator to (2.2) to calculate $\Delta_2 T(n)$ (and Proposition 3.3 for $\Delta_2 T(n_0-3)$). Use $\Delta_2 R(n) = 2 - \Delta_2 T(n-1)$ to calculate $\Delta_2 R(n)$. Use the definition of Δ_2 to calculate $R(n)$ and $T(n)$. ∎

Since $\Delta_2 U(n) := T(R(n)) - T(R(n-2))$, the preceding theorem shows that the reason $\Delta_2 U(n_0+1) = \Delta_2 U(n_0+2) = \Delta_2 U(n_0+3) = 0$ is not because the values of $T(n)$ at two distinct arguments of T happen to be equal, but rather because T is being evaluated at two equal values of $R(n)$, respectively, r_1, r_0, and r_1. For this reason, these three zero terms immediately following any two in $\{\Delta_2 U(n)\}$ are rather uninteresting. Note that these three zeros in each paeon of $\Delta_2 U(n)$ correspond precisely to those terms for which $\Delta_2 R(n) = 0$, or equivalently the ones for which $\varphi(n)$ is undefined.

Conversely when $\Delta_2 R(n) = 2$, that is, when $\varphi(n)$ is defined,

$$\Delta_2 U(n_0) = T(r_0) - T(r_0 - 2) = \Delta_2 T(r_0)$$
$$\Delta_2 U(n_0+4) = T(r_0+2) - T(r_0) = \Delta_2 T(r_0+2),$$

and so in both cases we obtain the key correspondence,

$$\Delta_2 U(n) = (\Delta_2 T)(R(n)) = T\big(R(n)\big) - T\big(R(n)-2\big). \tag{3.2}$$

We could also give an alternate definition: $\varphi(n) = P$ if $\Delta_2 U(n) + \Delta_2 R(n) = 4$, and $\varphi(n) = H$ if $\Delta_2 U(n) + \Delta_2 R(n) = 2$. Or also equivalently, $\varphi(n) = P$ if $\Delta_2 T(n) = 2$ and $\Delta_2 T(n-1) = 0$, and $\varphi(n) = H$ if $\Delta_2 T(n) = 0$ and $\Delta_2 T(n-1) = 0$.

Note 3.5. $\Delta_2 T(n)$, $\Delta_2 U(n)$, and $\Delta_2 R(n)$ exhibit a *recursive symmetry*, just as do many other meta-Fibonacci sequences in the literature (e.g., in [12, 10]). As a result, there is a natural partition of their domain into *generations* (finite, consecutive strings of increasing length) such that certain function values within one generation can be expressed elegantly in terms of function values in preceding generations. We show in Theorem 3.14 that (3.2) will express $\Delta_2 U$ in one generation in terms of $\Delta_2 T$ in the preceding generation.

DEFINITION 3.6 (generation). *For any* $g > 0$, *let* $m_g := \frac{1}{2}(3^{g+1}+5) = 3 + \sum_{i=0}^{g} 3^i$ *and call the interval* $[m_g, m_{g+1}-1]$ *the* gth generation, *written as* $\mathrm{gen}(g)$. *We partition the gth generation into two nonconsecutive subsequences: the* gth even semigeneration $\mathrm{sg}_0(g) := \{n \in \mathrm{gen}(g) \mid n \equiv m_g \pmod 2\}$ *and the* gth odd semigeneration $\mathrm{sg}_1(g) := \{n \in \mathrm{gen}(g) \mid n \not\equiv m_g \pmod 2\}$. *For any sequence* $s(n)$, *we will refer to the subsequence* $\{s(n) \mid n \in \mathrm{gen}(g),\ s(n) \text{ defined}\}$ *as the* gth generation of s *and similarly for semigenerations. An* even (odd) *foot is one that starts in an even (odd) semigeneration.*

Note that because the length 3^{g+1} of $\mathrm{gen}(g)$ is odd, m_g and m_{g+1} always have opposite parity. The foot that follows a paeon is always of the same parity, because the paeon has even length; while the foot that follows a hypercatalectic is always of opposite parity, because the hypercatalectic has odd length.

We list for future reference the first five values of m_g: 7, 16, 43, 124, 367.

Table 3.3
Function values over a line.

$n-n_0$	$T(n)$	$\Delta_2 T(n)$	$R(n)$	$\Delta_2 R(n)$	$\Delta_2 U(n)$	$\varphi(n)$
-2	t_0-2	?	r_0-2	0	0	–
-1	t_1-2	0	r_1	?	0	?
0	t_0	2	r_0	2	2	P
1	t_1	2	r_1	0	0	–
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	–
$4q-4$	t_0+4q-4	2	r_0+2q-2	2	2	P
$4q-3$	t_1+2q-2	2	r_1	0	0	–
$4q-2$	t_0+4q-2	2	r_0+2q-2	0	0	–
$4q-1$	t_1+2q-2	0	r_1	0	0	–
$4q$	t_0+4q-2	0	r_0+2q	2	0	H
$4q+1$	t_1+2q	2	r_1+2	2	2	P
$4q+2$	t_0+4q	2	r_0+2q	0	0	–

Now we refine further our description of the structure of $\Delta_2 U(n)$ within a generation. Once again we borrow from the terminology of Greek poetry, here to name sequences of feet with no internal hypercatalectics.

Definition 3.7 (line). *A* line *is a maximal subsequence of consecutive paeons and hypercatalectics in a generation, none of whose paeons is preceded by one of its hypercatalectics. We write a line as an overlined sequence of P's and H: e.g.,* $\overline{PPP\ldots PH}$. *An* even (odd) *line is one that starts in an even (odd) semigeneration, that is, one whose feet are all even (odd).*

Proposition 3.3 tells us that a line is either a nonempty sequence of paeons followed by a hypercatalectic or a sequence of paeons at the end of a generation.

Example 3.8. The first generation of $\Delta_2 U$ begins at $\Delta_2 U(7)$, ends at $\Delta_2 U(15)$, and consists of the 9 numbers $\{\Delta_2 U(n)\}_{n\in\text{gen}(1)} = \{2,0,0,0,\ 0,\ 2,0,0,0\}$, the 3 feet $\{\varphi(n)\}_{n\in\text{gen}(1)\cap\text{Dom}\,\varphi} = \{P,H,P\}$ that form the first generation of φ, or the 2 lines $\{\overline{PH}, \overline{P}\}$. The second generation of $\Delta_2 U$ continues on from $\Delta_2 U(16)$ and consists of the 27 numbers $\{\Delta_2 U(n)\}_{n\in\text{gen}(2)} = \{2,0,0,0,\ 2,0,0,0,\ 0,\ 2,0,0,0,\ 0,\ 2,0,0,0,\ 0,\ 2,0,0,0,\ 2,0,0,0\}$ or the 4 lines $\{\overline{PPH}, \overline{PH}, \overline{PH}, \overline{PP}\}$. The third generation of $\Delta_2 U$ consists of the 10 lines $\{\overline{PPPPH}, \overline{PH}, \overline{PH}, \overline{PH}, \overline{PPH}, \overline{PPH}, \overline{PH}, \overline{PH}, \overline{PH}, \overline{PPPP}\}$.

As was the case with feet, there is an intricate relationship among the values of the T, R, and U sequences on the paeons and hypercatalectic that make up a line. This is captured in the second of our three structural theorems.

Theorem 3.9 (line pattern theorem). *Suppose $T(n_0) = t_0$, $T(n_0+1) = t_1$, $R(n_0) = r_0$, $R(n_0+1) = r_1$, and $\Delta_2 U(n_0)$ is the beginning of a line of q paeons and a hypercatalectic, ending therefore at $\Delta_2 U(n_0+4q)$. Then $T(n)$, $\Delta_2 T(n)$, $R(n)$, $\Delta_2 R(n)$, $\Delta_2 U(n)$, and $\varphi(n)$ have the following values shown in Table* 3.3.

Proof. By Definition 3.7, $\Delta_2 U(n_0) = \Delta_2 U(n_0+4) = \cdots = \Delta_2 U(n_0+4q-4) = \Delta_2 U(n_0+4q+1) = 2$ and all other intervening values of $\Delta_2 U$ are zero. The first two rows of the table depend on whether or not the line begins a generation (and hence whether or not the preceding foot is a paeon or a hypercatalectic), and its values can be computed using Foot Pattern Theorem 3.4. The rest of the result follows from q applications of Foot Pattern Theorem 3.4, setting n_0 successively to the start of each of these paeons. ∎

We observe that the feet within a line all have the same parity, and the lines within a generation alternate in parity. Further, the lines within a generation are essentially symmetric about the middle of the generation; for example, in the third generation, the fifth and sixth are identical, as are the fourth and seventh, third and eighth, and second and ninth. The first and tenth lines have identical paeons, but the first line ends in a hypercatalectic while the tenth does not. In fact, it is precisely this symmetry that led us to the specification of m_g as the start of the gth generation.

We will show in Theorem 3.15, to which we are now building, that a generation consists entirely of lines, with no excess terms. Further, the proof of that theorem will demonstrate how the values of $\Delta_2 U(n)$ in the gth generation are determined by the values of $\Delta_2 T(n)$ in the $(g-1)$st generation. This provides an understanding of how successive generations interrelate. In order to get there, we require some additional background.

We mentioned in Note 3.5 that the gth generation of $\Delta_2 U$ would be expressed in terms of the $(g-1)$st generation of $\Delta_2 T$. It turns out that the natural restriction of the function R to those n in the gth generation that start a foot is one to one onto the $(g-1)$st generation; for each such n, the value of $R(n)$ is the *unique* r in the $(g-1)$st generation for which $\Delta_2 T(r) = \Delta_2 U(n)$. We can then use the inverse of this map to construct a beautiful correspondence from the $(g-1)$st generation to the gth, as we will soon show.

Example 3.10. We illustrate this inverse map of R with a simple example. Generation 2 begins at $n = 16$ and ends at $n = 42$. We know from Note 3.1 and Theorems 3.4 and 3.9 that the feet of $\Delta_2 U$ in generation 2 begin at $n = 16$, 20, 24, 25, 29, 30, 34, 35, and 39. From Table 3.1 we have $R(16) = 7$, $R(20) = 9$, $R(24) = 11$, $R(25) = 8$, $R(29) = 10$, $R(30) = 13$, $R(34) = 15$, $R(35) = 12$, and $R(39) = 14$. Observe that all of these values of $R(n)$ are distinct and that they include all of the integers from 7 to 15, which is precisely the first generation. So the inverse of R, evaluated on the first generation, gives the beginnings of all of the feet in the second generation of $\Delta_2 U$. In this way, the successive generations of T are interrelated.

We now show that the invertibility property holds.

Proposition 3.11. *Let E be the set of even numbers and $D = \operatorname{Dom}\varphi$. Then $R|_{D\cap E}$ begins with 5 and increases (strictly) only by 2, while $R|_{D\setminus E}$ begins with 4 and also increases only by 2.*

Proof. By (3.1) and Note 3.1, $R(7) = 4$, $R(12) = 5$. The result follows from the values of $R(n_0 + 4)$ and $R(n_0 + 5)$ given in Foot Pattern Theorem 3.4. ∎

We do not know yet that $\operatorname{Ran}(R|_{\operatorname{Dom}\varphi})$ is all of $[4, \infty)$. This will be established after we show in Theorem 3.15 that each generation of $\Delta_2 U(n)$ has at least one hypercatalectic, so $\operatorname{Ran}(R|_{\operatorname{Dom}\varphi})$ flips parity infinitely often. We now name the inverse of this restriction of R.

Definition 3.12. *For $n \in \operatorname{Dom}\varphi$, let $f(R(n)) = n$. That is, if $\Delta_2 R(n) = 2$, then $f(R(n)) = n$.*

In Example 3.10, we had $f(7) = 16$, $f(8) = 25$, $f(9) = 20$, and so on. Also, since f is bijective by construction, $R(f(r)) = r$ when $f(r)$ is defined. Observe that $R(f(7)) = R(16) = 7$ and $f(7) = 16$ and $R(7) = 4$ so $f(7) \equiv R(7) \not\equiv 7 \pmod 2$.

Corollary 3.13. *When $f(r)$ is defined, $f(r) \equiv R(r) \not\equiv r \pmod 2$. If $r \in \operatorname{Ran} R$, then $f(n)$ is defined for all $n \equiv r \pmod 2$ in $[4, r]$. $f(r)$ is the smallest member of the set $R^{-1}(r)$.*

Proof. By Proposition 3.11, $R(n)$ is even when n is odd, and vice versa. Since f

is the inverse of a restriction of R, $f(r)$ is odd when r is even, and vice versa.

Also by Proposition 3.11, the range of $R|_{D\cap E}$ is a consecutive sequence of odd numbers starting at 5 with no gaps; so if an odd $r \in \operatorname{Ran} R$, then $\operatorname{Ran} R$ also includes all lesser odd numbers down to 5; and similarly for even r.

By Theorem 3.4, $\Delta_2 R(n)$ is 2 at the beginning of a foot in $\Delta_2 U$ and 0 elsewhere. Therefore, among all the n for which $R(n)$ is equal to a particular r, it is the smallest one which belongs to $\operatorname{Dom} \varphi$. □

We are now close to demonstrating that our partition of the domains of our functions into generations is a natural one, by showing that the values of functions on one generation depend solely on the values of functions in the immediately preceding generation. As we illustrated in Example 3.10, if n belongs to the gth generation, then $R(n)$ belongs to the $(g-1)$st generation. Therefore, if r belongs to the gth generation, then $f(r)$ belongs to the $(g+1)$st generation.

It turns out though that f does much more than simply map one generation into the next. If for some r in the gth generation $\Delta_2 T(r) = 2$, then $f(r)$ is the beginning of a paeon in the $(g+1)$st generation of $\Delta_2 U$: that is, $\varphi(f(r)) = P$. Conversely if $\Delta_2 T(r) = 0$, then $f(r)$ is the beginning of a hypercatalectic of $\Delta_2 U$ and $\varphi(f(r)) = H$. Thus f establishes a correspondence between each member of one generation and each *foot* of $\Delta_2 U$ in the next generation, incidentally accounting for how the successive generations grow.

Figure 3.1 displays this correspondence for the first two generations and elaborates on the consequences of the parity principle of Corollary 3.13. As shown in Example 3.8, the first generation of $\Delta_2 U$ consists of the feet $\{P, H, P\}$. According to Foot Pattern Theorem 3.4, it follows that the first generation of $\Delta_2 T$ must be $\{2,2,2,0;0;2,2,2,0\}$, where semicolons indicate the ends of the feet in $\Delta_2 U$. In Figure 3.1 we show the four entries of $\Delta_2 T$ that correspond to each paeon of $\Delta_2 U$ enclosed in a parallelogram, and the single (zero) entry of $\Delta_2 T$ that corresponds to a hypercatalectic of $\Delta_2 U$ enclosed in a triangle.

The action of f on members of the first generation determines the structure of the second generation. If we have r such that $\Delta_2 T(r) = 2$, then $f(r)$ marks the beginning of a paeon in $\Delta_2 U$: that is, $\varphi(f(r)) = P$, as indicated by the dashed lines in the figure. Likewise dashed lines connect the points at which $\Delta_2 T(r) = 0$ to the points where $\varphi(f(r)) = H$.

Successive integers in the first generation alternate in parity. By Corollary 3.13, their images under f must also alternate in parity. However, since all of the feet in a line must have the same parity, the image under f of the integers corresponding to a paeon in $\Delta_2 U$ lie in two separate lines of opposite parity. The lines in the second generation are enclosed in boxes, and arrows indicate the intertwined order in which they are to be concatenated to form the second generation.

Observe that we can now easily count the number of feet and lines in each successive generation. Since each foot in the first generation of $\Delta_2 U$ ends at a point where $\Delta_2 T$ is zero, it corresponds to a line-ending hypercatalectic in the second generation of $\Delta_2 U$. So, the number of hypercatalectics in one generation is the same as the number of feet in the preceding generation. And hence, the number of lines in one generation is one more (taking into account the last line that lacks a hypercatalectic) than the number of feet in the preceding generation.

The following theorem proves the correspondence between 0's and 2's and H's and P's and will be the foundation for much of the rest of the paper.

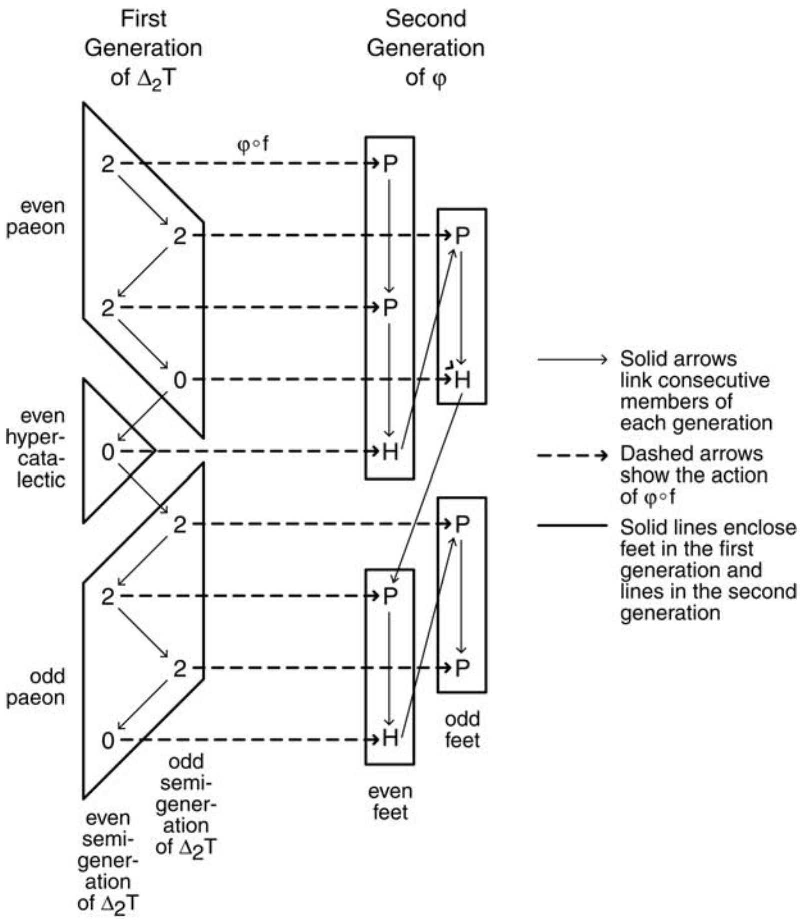


FIG. 3.1. *How successive generations correspond.*

THEOREM 3.14 (generational correspondence theorem). *The diagram*

$$\begin{array}{ccc} \operatorname{Dom} f & \underset{R}{\overset{f}{\rightleftarrows}} & \operatorname{Ran} f \\ {\scriptstyle \Delta_2 T}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \{0,2\} & \underset{2\mapsto P}{\overset{0\mapsto H}{\longrightarrow}} & \{H,P\} \end{array}$$

commutes. That is, for $4 \le r \in \operatorname{Dom} f$, $\varphi(f(r)) = P$ *iff* $\Delta_2 T(r) = 2$.

Proof. If $\varphi(f(r)) = P$, then by (3.2), $2 = \Delta_2 U(f(r)) = \Delta_2 T(R(f(r))) = \Delta_2 T(r)$. If $\varphi(f(r)) = H$, then $0 = \Delta_2 U(f(r)) = \Delta_2 T(R(f(r))) = \Delta_2 T(r)$. ∎

We are now ready to present the last of our three structural theorems, which will rely on Theorem 3.14 to express each generation in terms of its predecessor. This will prove that the correspondence illustrated in Figure 3.1 does map each generation exactly onto the next.

THEOREM 3.15 (generation pattern theorem). *The gth generation of* $\Delta_2 U$ *consists entirely of* $3^{g-1}+1$ *lines. Its last line consists only of (odd) paeons, the last of which has Foot Pattern Theorem* 3.4 *parameters* $n_0 = m_{g+1} - 4$, $t_0 := T(n_0) = 3^{g+1} - 2$, $t_1 := T(n_0+1) = 3^{g+1}$, $r_0 := R(n_0) = m_g - 2$, *and* $r_1 := R(n_0+1) = m_g - 1$.

Table 3.4
Function values near the end of a generation.

n	$T(n)$	$\Delta_2 T(n)$	$R(n)$	$\Delta_2 R(n)$	$\Delta_2 U(n)$	$\varphi(n)$
$m_g - 4$	$3^g - 2$	2	$m_{g-1} - 2$	2	2	P
$m_g - 3$	3^g	2	$m_{g-1} - 1$	0	0	–
$m_g - 2$	3^g	2	$m_{g-1} - 2$	0	0	–
$m_g - 1$	3^g	0	$m_{g-1} - 1$	0	0	–
...						
m_g	$3^g + 2$	2	m_{g-1}	2	2	P
$m_g + 1$	$3^g + 2$	2	$m_{g-1} - 1$	0	0	–
$m_g + 2$	$3^g + 4$	2	m_{g-1}	0	0	–

Proof. The values of T and $\Delta_2 U$ necessary to verify the result for $g = 1$ and $g = 2$ have been listed above in Note 3.1. We proceed to larger g by induction, assuming that the results hold for all generations before the gth. In particular, applying Foot Pattern Theorem 3.4 to the last foot of the $(g-1)$st generation gives us the first four rows of the key function values shown in Table 3.4.

To obtain the last three of the rows, we need to show that the gth generation begins with a paeon and not a hypercatalectic. Observe that by induction the $(g-1)$st generation begins with a paeon, so the Foot Pattern Theorem 3.4 applied to $n_0 = m_{g-1} - 4$ gives us $T(m_{g-1}) = 3^{g-1} + 2$ and $\Delta_2 T(m_{g-1}) = 2$. Since $\Delta_2 U(m_g)$ begins a foot, (3.2) tells us that $\Delta_2 U(m_g) = \Delta_2 T(m_{g-1}) = 2$ and the start of the first foot in the gth generation of $\Delta_2 U$ corresponds with the first term in the $(g-1)$st generation of $\Delta_2 T$, as desired. Apply Theorem 3.4 to $n_0 = m_g$ to fill in the rest of the values.

Next, we show that f maps $\mathrm{gen}(g-1)$ *onto* $\mathrm{gen}(g) \cap \mathrm{Dom}\,\varphi$, as shown for $g = 2$ in Figure 3.1. By Corollary 3.13 and the fact that all generations have odd length, we will know then that f maps $\mathrm{sg}_i(g-1)$ onto $\mathrm{sg}_i(g) \cap \mathrm{Dom}\,\varphi$ for $i = 0, 1$.

We begin at $R(m_g) = m_{g-1}$, so $f(m_{g-1}) = m_g$, for which Theorem 3.14 with $r = m_{g-1}$ gives us a correspondence between $\Delta_2 T(m_{g-1}) = 2$ and $\varphi(f(m_{g-1})) = P$. We claim that if we continue on through all of the $(g-1)$st generation of $\Delta_2 T$, the values of f will not overrun the end of the gth generation. To check this, we have to carefully count 2's, 0's, paeons, and hypercatalectics.

We look first at the 2's and 0's in the $(g-1)$st generation of $\Delta_2 T$. In the even case, $\sum_{n \in \mathrm{sg}_0(g-1)} \Delta_2 T(n) = T(m_g - 1) - T(m_{g-1} - 2) = 3^g - 3^{g-1} = 2 \cdot 3^{g-1}$ and $(m_g - 1) - (m_{g-1} - 2) = 3^g + 1$, so $\Delta_2 T(n)$ has the value two 3^{g-1} times and zero the remaining $\frac{1}{2}(3^g + 1) - 3^{g-1} = \frac{1}{2}(3^{g-1} + 1)$ times. In the odd case, $\sum_{n \in \mathrm{sg}_1(g-1)} \Delta_2 T(n) = T(m_g - 2) - T(m_{g-1} - 1) = 3^g - 3^{g-1} = 2 \cdot 3^{g-1}$ and $(m_g - 2) - (m_{g-1} - 1) = 3^g - 1$, so $\Delta_2 T(n)$ is two 3^{g-1} times and zero the remaining $\frac{1}{2}(3^g - 1) - 3^{g-1} = \frac{1}{2}(3^{g-1} - 1)$ times.

Using the correspondence of Theorem 3.14, we conclude that in the gth generation the $\frac{1}{2}(3^{g-1}+1)$ even hypercatalectics outnumber the $\frac{1}{2}(3^{g-1}-1)$ odd hypercatalectics by one. Thus, the correspondence ends (after a total of $3^{g-1} + 1$ lines) with an even hypercatalectic followed by a run of odd paeons. There are 3^{g-1} even paeons and 3^{g-1} odd paeons. Adding together the lengths of all the paeons and hypercatalectics gives $4 \cdot 3^{g-1} + 4 \cdot 3^{g-1} + \frac{1}{2}(3^{g-1}+1) + \frac{1}{2}(3^{g-1}-1) = 3^{g+1}$, which is exactly the length of the gth generation.

Finally, we need to calculate the Foot Pattern Theorem 3.4 parameters when $n_0 = m_{g+1} - 4$ at the end of the gth generation. Let $P_0 = P_1 = 3^{g-1}$, $H_0 = \frac{1}{2}(3^{g-1} + 1)$, and $H_1 = \frac{1}{2}(3^{g-1} - 1)$ be, respectively, the number of even paeons, odd paeons, even hypercatalectics, and odd hypercatalectics in the gth generation. Then by repeated application of Line Pattern Theorem 3.9, $r_0 = (m_{g-1} - 1) + 2H_0 + 2(P_1 - 1) = m_{g-1} - 1 + 3^{g-1} + 1 + 2 \cdot 3^{g-1} - 2 = m_{g-1} + 3^g - 2 = m_g - 2$, $r_1 = m_{g-1} + 2H_1 + 2P_1 = m_g - 1$, $t_0 = (3^g + 2) + 4P_0 + 2P_1 - 4 = 3^{g+1} - 2$, and $t_1 = (3^g + 2) + 4P_0 + 2P_1 - 2 = 3^{g+1}$. ∎

4. Some interesting properties of the three-term case. We continue to use T, U, and R as defined in (3.1). In this section we explore a variety of interesting properties of the T sequence, as well as of the $\Delta_2 T$ and $\Delta_2 U$ difference sequences.

Before we proceed, we need to present a short technical lemma concerning the relationship between differences in the T sequence and differences in the U sequence.

LEMMA 4.1. $\Delta_d T(n) = \sum_{i=0}^{k-1} \Delta_d U(n-i) = \sum_{i=0}^{d-1} \Delta_k U(n-i)$.

Proof. Rearrange the summation and difference operators as follows:

$$\begin{aligned}\Delta_d T(n) &= \sum_{i=0}^{k-1} \Delta_d U(n-i) = \sum_{i=0}^{k-1} U(n-i) - \sum_{i=0}^{k-1} U(n-d-i) \\ &= \sum_{i=0}^{k+d-1} U(n-i) - \sum_{i=0}^{d-1} U(n-k-i) - \sum_{i=0}^{k-1} U(n-d-i) \\ &= \sum_{i=0}^{d-1} U(n-i) - \sum_{i=0}^{d-1} U(n-k-i) = \sum_{i=0}^{d-1} \Delta_k U(n-i). \qquad \square\end{aligned}$$

Allenby and Smith [1] first observed that there is no n for which $T(n) = T(n-2) = T(n-4)$, or equivalently, for which $\Delta_2 T(n) = \Delta_2 T(n-2) = 0$. This follows directly from Foot Pattern Theorem 3.4.

PROPOSITION 4.2. *There do not exist n for which $\Delta_2 T(n) = \Delta_2 T(n-2) = 0$.*

Proof. The proof follows directly from Foot Pattern Theorem 3.4. ∎

In fact, Proposition 4.2 is equivalent to Proposition 3.3: if $\Delta_2 T(n) = \Delta_2 T(n-2) = 0$, then by Corollary 2.3, $\Delta_2 U(n) = \Delta_2 U(n-1) = \cdots = \Delta_2 U(n-4) = 0$; while if Proposition 4.2 holds, then for any n, $0 \neq \Delta_4 T(n) = \Delta_2 T(n) + \Delta_2 T(n-2) = \Delta_2 U(n) + \Delta_2 U(n-1) + 2\Delta_2 U(n-2) + \Delta_2 U(n-3) + \Delta_2 U(n-4)$, so $\Delta_2 U$ never has five consecutive zeros.

One of the more interesting properties of Conolly's sequence (1.2), our case $k = 2$, is that it consists of a monotonically increasing sequence of integers whose frequency counts form the Gray binary sequence (Sloane Sequence A001511 [13]), omitting its first term. We can prove the analogous property concerning the frequency counts of $T(n)$ for $k = 3$ and in so doing answer a conjecture given in Allenby and Smith [1].

THEOREM 4.3. $T(n) = T(n+1) = T(n+2)$ *iff* $n = m_g - 3$ *for some* g.

Proof (First conjectured in Allenby and Smith [1]). The Foot Pattern Theorem 3.4 parameters proven in Generation Pattern Theorem 3.15 give us the required equality when $n = m_g - 3$. We need to show then that equality does not occur elsewhere.

Suppose n^* is a minimal counterexample. That is, $n^* + 3$ is not the start of a generation, $t := T(n^*) = T(n^* + 1) = T(n^* + 2)$, and no smaller counterexamples exist. By Theorem 3.4, $\varphi(n^* - 1) = P$, $R(n^* + 1) = R(n^* - 1) = n^* - t + 1$, and $R(n^* + 2) = R(n^*) = R(n^* - 2) = n^* - t + 2$. By Lemma 4.1, $0 = \Delta_1 T(n^* + 2) = \Delta_3 U(n^* + 2) = T(n^* - t + 2) - T(n^* - t + 1) = \Delta_1 T(n^* - t + 2)$.

By Theorem 3.4, Lemma 4.1, and (3.1), we have $2 = \Delta_2 U(n^* - 1) = (\Delta_2 T)(R(n^* - 1)) = (\Delta_2 T)(R(n^* + 1)) = (\Delta_2 U)(R(n^* + 1)) + (\Delta_2 U)(R(n^* + 1) - 1) + (\Delta_2 U)(R(n^* +

$1) - 2) = \Delta_2 U(n^* - t + 1) + \Delta_2 U(n^* - t) + \Delta_2 U(n^* - t - 1)$. Exactly one of these three terms is equal to 2: but which?

If $\Delta_2 U(n^* - t - 1) = 2$, then by Theorem 3.4, $T(n - t) = T(n - t + 2)$; and since we have already observed that $T(n - t + 1) = T(n - t + 2)$, we can invoke the minimality of n^* to conclude that there must be a g such that $m_g = n^* - t + 3$. But then $R(n^* - 1) = m_g - 2$ and $n^* - 1 = f(m_g - 2) = m_{g+1} - 4$, so $n^* = m_{g+1} - 3$, in contradiction to our hypothesis.

If $\Delta_2 U(n^* - t + 1) = 2$, then there must be a g such that $m_g = n^* - t + 1$ and similarly $n^* = m_{g+1} + 1$, which contradicts the observation in Generation Pattern Theorem 3.15 that the second and third members of any generation of T differ.

Finally, if $\Delta_2 U(n^* - t) = 2$, then there must be a g such that $m_g = n^* - t + 4$. $R(m_{g+1} - 6) = m_g - 4$ and $R(m_{g+1}) = m_g$, so by monotonicity R attains the value $m_g - 2$ at most twice, which contradicts $R(n^* - 2) = R(n^*) = R(n^* + 2) = m_g - 2$.

So none of the three choices is possible, and the original assumption of a counterexample must have been false. □

COROLLARY 4.4. *There is no n for which $T(n) = T(n+1) = T(n+2) = T(n+3)$.*

Proof. This is proven directly in Allenby and Smith [1] but follows simply from Proposition 4.3. □

In fact, it turns out that even two consecutive terms of T are rarely equal. Looking back at Figure 1.6, we can see that the points at which the two lines intersect all occur near the ends of generations.

COROLLARY 4.5. *$T(n) = T(n + 1)$ iff there is a g such that $n \in \{m_g - 5, m_g - 3, m_g - 2, m_g\}$.*

Proof. If $n \in \{m_g - 5, m_g - 3, m_g - 2, m_g\}$, then by the Generation Pattern Theorem 3.15, $T(n) = T(n + 1)$. Conversely, suppose $T(n) = T(n - 1)$ and consider where the nearest paeon starts. If $\varphi(n) = P$, then by Foot Pattern Theorem 3.4, $T(n - 3) = T(n - 2) = T(n - 1) = T(n) - 2$ and by Theorem 4.3, $n = m_g$ for some g. If $\varphi(n + 1) = P$, then similarly $n = m_g - 5$. If $\varphi(n + 2) = P$, then $n = m_g - 2$. Otherwise, $T(n) = T(n + 1) = T(n + 2)$ and $n = m_g - 3$. □

COROLLARY 4.6. *There are infinitely many n for which $T(n) = T(n + 1)$ and $T(n + 2) = T(n + 3)$.*

Proof. This was conjectured in Allenby and Smith [1] and follows from Corollary 4.5. The values of n for which this holds are precisely those that can be written as $n = m_g - 2$ for some g. It is shown in [1] that such n must satisfy $\{\Delta_1 T(n-1), \Delta_1 T(n+5)\} = \{-2, 2\}$ (in some order), which follows from the Generation Pattern Theorem 3.15. This theorem also tells us that if $T(n) = T(n+1)$, then either $T(n-2) = T(n-1)$ or $T(n + 2) = T(n + 3)$. □

COROLLARY 4.7. *There is no n for which $T(n) = T(n+1)$, $T(n+2) = T(n+3)$, and $T(n + 4) = T(n + 5)$.*

Proof. This is proven directly in Allenby and Smith [1] but follows simply from Corollary 4.5. □

Theorem 4.3 and the four corollaries that follow from it provide the characterization of the frequency counts that we seek. Because the case $k = 3$ gives a nonmonotonic sequence (unlike Conolly's $k = 2$), its repeated values can be either consecutive or nonconsecutive. We define the *consecutive repeated values* of a sequence

$$\{\overbrace{a_1, \ldots, a_1}^{n_1}, \overbrace{a_2, \ldots, a_2}^{n_2}, \ldots, \overbrace{a_m, \ldots, a_m}^{n_m}\},$$

where $a_1 \neq a_2$, $a_2 \neq a_3$, $\ldots$, $a_{m-1} \neq a_m$ in the obvious way as the sequence $\{n_1, \ldots, n_m\}$. Then we can summarize these results in the following proposition.

PROPOSITION 4.8. *The consecutive repeated values of the gth generation of T are*

$$\{2, \overbrace{1, \ldots, 1}^{3^{g+1}-7}, 2, 3\}.$$

Proof. This is a restatement of Corollary 4.5. ∎

In section 3 we showed the crucial role played by the $\Delta_2 U$ sequence that leads to our understanding of the detailed structure of the T sequence. It turns out that the $\Delta_2 U$ sequence has many fascinating properties, including beautiful symmetries both within and between generations, that make it of independent interest.

We begin by giving a more detailed description of the structure of each generation of $\Delta_2 U$. By Generation Pattern Theorem 3.15, the gth generation of $\Delta_2 U$ consists entirely of $3^{g-1} + 1$ lines.

DEFINITION 4.9. *For $g > 0$ and $i \in [0, 3^{g-1}]$ let $q_{i,g}$ be the number of paeons in the ith line (numbered starting with the zeroth) of the gth generation of $\Delta_2 U$.*

In Example 3.8 we showed that $q_{0,1} = q_{1,1} = 1$; $q_{0,2} = 2$, $q_{1,2} = q_{2,2} = 1$, $q_{3,2} = 2$; $q_{0,3} = 4$, $q_{1,3} = q_{2,3} = q_{3,3} = 1$, $q_{4,3} = q_{5,3} = 2$, $q_{6,3} = q_{7,3} = q_{8,3} = 1$, and $q_{9,3} = 4$.

PROPOSITION 4.10. *The sequence consisting of the number of paeons in each even line in the $(g+1)$st generation of $\Delta_2 U$ can be expressed in terms of the number of paeons in the lines in the gth generation of $\Delta_2 U$ as follows:*

$$\{\, 2q_{0,g}, \overbrace{1, \ldots, 1}^{q_{1,g}}, 2q_{2,g}, \overbrace{1, \ldots, 1}^{q_{3,g}}, 2q_{4,g}, \overbrace{1, \ldots, 1}^{q_{5,g}}, \ldots, 2q_{3^{g-1}-1,g}, \overbrace{1, \ldots, 1}^{q_{3^{g-1},g}} \}.$$

Proof. Generational Correspondence Theorem 3.14 (see Figure 3.1), together with Generation Pattern Theorem 3.15, shows that each even (odd) paeon in a generation produces two paeons in the next even (odd) semigeneration and a paeon and a hypercatalectic in the next odd (even) semigeneration, while an even (odd) hypercatalectic simply produces an even (odd) hypercatalectic. Thus each paeon contributes a single paeon line (i.e., PH) to the next subgeneration of opposite parity and two paeons (PP) to the next subgeneration of equal parity. The number of even paeons in each even line in the gth generation therefore appears doubled in the $(g+1)$st even semigeneration, alternating with runs of single-paeon lines. ∎

Recall that a sequence is *palindromic* if it has reflective symmetry. Two basic transformations that preserve palindromicity are that of replacing every occurrence of a member of a sequence by a palindromic subsequence, and that of applying any transformation to the lengths of consecutive runs of identical members in a sequence. For example, the string "AABABAA" is palindromic and changing every "AA" to "A" and every "B" to "CDC" gives "ACDCACDCA," which remains palindromic. We use these palindromicity-preserving transformations and Foot Pattern Theorem 3.4 to show the following generational palindromicity property, which we can use to find $q_{i,g}$ for odd i.

THEOREM 4.11. *Each generation of $\Delta_2 U(n)$ consists of a palindromic sequence of feet.*

Proof. The result is true for the first generation, which consists of $\{P, H, P\}$; see Example 3.8. We proceed by induction, assuming that it is true for all generations before the gth.

By induction, the $(g-1)$st generation of $\Delta_2 U$ is a palindromic sequence of feet. Transforming that sequence using $\{P \mapsto 02220, H \mapsto 0\}$ followed by $\{000 \mapsto 00, 00 \mapsto$

$0\}$ gives us a palindromic sequence, which Foot Pattern Theorem 3.4 lets us recognize as the $(g-1)$st generation of $\Delta_2 T$, preceded by a zero. Because the length of every generation is odd, it follows that the $(g-1)$st even semigeneration of $\Delta_2 T(n)$ is the reverse of the $(g-1)$st odd semigeneration of $\Delta_2 T(n)$, followed by a zero. Then by Generational Correspondence Theorem 3.14 (see Figure 3.1), the sequence $\{q_{0,g}, q_{2,g}, q_{4,g}, \dots\}$ is the reverse of $\{q_{1,g}, q_{3,g}, q_{5,g}, \dots\}$. Thus, the entire sequence $\{q_{i,g}\}_{i=0}^{3^g}$ is palindromic and all the feet in the gth generation form a palindromic sequence. ∎

COROLLARY 4.12. *Each generation of* $\Delta_2 T(n)$ *consists of a palindromic sequence of* 2*'s and* 0*'s, followed by an extra* 0.

Proof. The proof follows from Theorem 4.11 by application of Foot Pattern Theorem 3.4. ∎

PROPOSITION 4.13. $q_{i,g}$ *is always a power of* 2.

Proof. The proposition is true for $g = 1$, since $q_{0,1} = q_{1,1} = 1$. Assume inductively that the proposition is true for all generations preceding the gth. Proposition 4.10 tells us that $q_{i,g}$ for even i is a power of 2, and Theorem 4.11 tells us that the $q_{i,g}$ for odd i have the same (power of 2) values in reverse order. ∎

We now use what we have learned about the structure of our sequences to obtain some quantitative properties of the sequences.

PROPOSITION 4.14. *For positive* g *and* $0 \le x \le m_{g+1} - 4$*, the sum* $T(m_g + x) + T(m_{g+1} - 4 - x) = 3^g + 3^{g+1}$ *is constant. For* $0 \le y \le m_{g+1} - 6$*, the sum* $U(m_g + y) + U(m_{g+1} - 6 - y) = 3^{g-1} + 3^g$ *is constant.*

Proof. The proof follows readily from palindromicity of $\Delta_2 T(n)$ and $\Delta_2 U(n)$ within the gth generation. ∎

PROPOSITION 4.15. *The mean value of the gth generation of* $T(n)$ *is* $2 \cdot 3^g + 1$*. The mean value of the gth generation of* $U(n)$ *is* $2 \cdot 3^{g-1} + \frac{5}{9}$.

Proof. The proof follows from Proposition 4.14 and the values indicated by Generation Pattern Theorem 3.15 and Foot Pattern Theorem 3.4 for the last few values in the gth generations of $T(n)$ and $U(n)$. ∎

COROLLARY 4.16. *The asymptotic value of* $\frac{T(n)}{n}$ *is* $\frac{2}{3}$.

Proof. The mean value of n in the gth generation is $(m_g + m_{g+1} - 1)/2 = (3^{g+1} + 5 + 3^{g+2} + 5 - 2)/4 = 3^{g+1} + 2 = 3 \cdot 3^g + 2$. By Proposition 4.15, the mean value of T in the same generation is $2 \cdot 3^g + 1$. The ratio $(2 \cdot 3^g + 1)/(3 \cdot 3^g + 2)$ approaches $\frac{2}{3}$ as g approaches infinity. ∎

PROPOSITION 4.17. *The mean value of the gth generation of* $\Delta_2 T(n)$ *is* $\frac{4}{3}$*. The mean value of the gth generation of* $\Delta_2 U(n)$ *is* $\frac{4}{9}$.

Proof. In the proof of the Generation Pattern Theorem 3.15, we counted $2 \cdot 3^{g-1}$ paeons in the gth generation, which has length 3^{g+1}. Recall from Foot Pattern Theorem 3.4 that at a hypercatalectic, $\Delta_2 T(n) = \Delta_2 U(n) = 0$, while each paeon contributes 6 to the sum of $\Delta_2 T$ and 2 to the sum of $\Delta_2 U$. ∎

We conclude this section with an observation whose generalization will play a key role in the next section.

PROPOSITION 4.18. $\Delta_4 U(n) \in \{0, 2\}$.

Proof. The result is easily verified for small n. Expand $\Delta_6 U(n)$ in two ways to obtain $\Delta_4 U(n) = \Delta_4 U(n-2) + (\Delta_2 U(n) - \Delta_2 U(n-4))$. By Foot Pattern Theorem 3.4, $\Delta_6 U$ has period 2 (alternating between the values 2 and 0) on any run of paeons and is 0 at a hypercatalectic. ∎

To date, we have not yet succeeded in proving that no closed form exists for $T(n)$. Nonetheless, the tools that we have created have facilitated the analysis which normally follows from the discovery of a closed form, namely, a complete characterization of the structure of the sequence, the computation of mean and asymptotic values, and the rapid calculation of large values of the sequence.

5. The structure for general odd *k*. In the preceding two sections, we examined the case $k = 3$ in great detail. In this section, we find that most of our results generalize to greater odd k. In the recursion (1.6) we again set $a = 0$ and assume the initial conditions $T_{0,k}(n) = 1$ for $1 \le n \le k$ with k odd. We will refer to this sequence as $T(n)$, to $R_{0,k}(n)$ as $R(n)$, and to $U_{0,k}(n)$ as $U(n)$:

$$\begin{aligned} T(n) &:= U(n) + U(n-1) + \cdots + U(n-k+1), \quad n > k, \\ U(n) &:= T(R(n)), \quad n > 1, \\ R(n) &:= n - T(n-1), \quad n > 1. \end{aligned} \tag{5.1}$$

When $k = 3$, the behavior of the recursion can largely be characterized in terms of the pattern of feet in the values of $\Delta_{k-1}U$, most of which have length $k + 1 = 4$, with the occasional hypercatalectic. For general odd k, we find that $\Delta_{k-1}U$ follows the same pattern, consisting mostly of paeons now of length $k+1$, with a hypercatalectic of length 1 every now and then.

In order to prove that there is only one zero or hypercatalectic between runs of paeons in $\Delta_{k-1}U$—which is not the case for even k—we will need to prove an additional property of these runs of paeons. This is that paeons always come in groups of $\frac{k-1}{2}$, a property which was trivial for $k = 3$, where $\frac{k-1}{2} = 1$.

It happens that $\Delta_{k+1}U$ has repeating patterns resembling feet of length $k - 1$, also with the occasional extra term resembling a hypercatalectic. We will show that the hypercatalectics in $\Delta_{k-1}U$ *always coincide* with the analogous disruptions in $\Delta_{k+1}U$. This gives the recursion an additional level of structure: paeons always come grouped in *polypaeons* of length $\operatorname{lcm}(k-1, k+1) = \frac{1}{2}(k^2 - 1)$. That is, a polypaeon consists of $\frac{k-1}{2}$ consecutive paeons.

We begin by generalizing our previous definitions and defining what we mean by a polypaeon.

Definition 5.1. *A* paeon *is a sequence*

$$\{k-1, \overbrace{0, \ldots, 0}^{k}\}$$

of $k+1$ consecutive values of $\Delta_{k-1}U(n)$. A hypercatalectic *is a singleton sequence $\{0\}$, immediately preceded in $\{\Delta_{k-1}U(n)\}$ by a paeon. A* foot *is either a paeon or a hypercatalectic. A* polypaeon *is a sequence of $\frac{1}{2}(k-1)$ consecutive paeons. For convenience, we will write $\{P\}$ interchangeably with the paeon $\{k-1, 0, \ldots, 0\}$ and likewise $\{H\}$ with the hypercatalectic $\{0\}$ when listing values of $\Delta_{k-1}U$. We also define $\varphi(n)$ on a subset of the natural numbers as the symbol P if $\Delta_{k-1}U(n) = k-1$ begins a paeon and H if $\Delta_{k-1}U(n) = 0$ is a hypercatalectic, and we leave it undefined otherwise.*

We now generalize Proposition 2.2 to facilitate proving the polypaeon structure.

Proposition 5.2. *For fixed odd $k > 1$, all even $d \in [0, k-1]$, and any $n > k$, both $|\Delta_d U(n)|$ and $\Delta_d T(n)$ belong to the set $\{0, k-1\}$.*

Proof. The case $k = 3$ was already proven in Proposition 2.2, and the case $d = 0$ is trivial. We assume therefore in what follows that $k > 3$ and $d > 0$.

As in the proof of Proposition 2.2, we proceed by induction and leave it to the reader to verify that the result holds for $n \le 2k$. Assume then that $n > 2k$ and that the result holds for lesser n and all relevant d.

$\Delta_d U(n) := T(R(n)) - T(R(n-d)) = (\Delta_x T)(R(n))$, where $x = \Delta_d R(n) = d - \Delta_d T(n-1)$. By the induction assumption, $\Delta_d T(n-1) \in \{0, k-1\}$, so $x \in \{d, d-k+1\}$. In either case, $|x|$ is even and in $[0, k-1]$ and the greater of $R(n)$ and $R(n-d)$ is strictly less than n, so we can apply induction again to find that $|(\Delta_x T)(R(n))| \in \{0, k-1\}$ and hence $|\Delta_d U(n)| \in \{0, k-1\}$.

Observe that $\Delta_d T(n) + \Delta_{k-1-d} T(n-d) = \Delta_{k-1} T(n)$. By Proposition 2.2, $\Delta_{k-1} T(n) \in \{0, k-1\}$; and by induction, $\Delta_{k-1-d} T(n-d) \in \{0, k-1\}$. Therefore, $|\Delta_d T(n)| \in \{0, k-1\}$. We need to rule out the case $\Delta_d T(n) = 1-k$. We expand $\Delta_d T(n)$ as a telescoping sum:

$$\Delta_d T(n) = \sum_{\substack{i \text{ even} \\ \in [0,d)}} \Delta_2 T(n-i). \tag{5.2}$$

By induction, when $i > 0$, $\Delta_2 T(n-i) \in \{0, k-1\}$. To prove that $\Delta_d T(n) \ge 0$, it therefore suffices to show that the first summand $\Delta_2 T(n) \ge 0$, that is, $\Delta_2 T(n) \ne 1-k$. Write

$$\Delta_2 T(n) = \Delta_{k-1} T(n) - \Delta_{k-3} T(n-2). \tag{5.3}$$

We will assume that $\Delta_2 T(n) = 1-k$ and show that a contradiction ensues. From (5.3) it follows that $0 = \Delta_{k-1} T(n)$ and $k-1 = \Delta_{k-3} T(n-2)$. But $\Delta_{k-3} T(n-2) = \Delta_{k-1} T(n-2) - \Delta_2 T(n-k+1)$, and by the same reasoning $0 = \Delta_2 T(n-k+1)$, which will be contradicted below.

By Lemma 4.1, $1-k = \Delta_2 T(n) = \Delta_k U(n) + \Delta_k U(n-1) = \Delta_{k+1} U(n) + \Delta_{k-1} U(n-1) = \Delta_2 U(n) + \Delta_{k-1} U(n-2) + \Delta_{k-1} U(n-1)$. Since by Proposition 2.2 the last two terms are nonnegative, $\Delta_2 U(n) = 1-k$. $\Delta_2 R(n) = 2 - \Delta_2 T(n-1) \in \{2, 3-k\}$ by induction, but if it were 2, then $\Delta_2 U(n)$ would be equal to $(\Delta_2 T)(R(n)) \in \{0, k-1\}$, contradicting our earlier assumption. Thus $\Delta_2 T(n-1) = k-1$ and $\Delta_2 R(n) = 3-k$. By (5.2), $\Delta_{k-1} T(n-1) = k-1$.

So $\Delta_{k-1} T(n-1) - \Delta_{k-1} T(n) = (k-1) - 0$, and if we expand this using (5.1) and gather terms, $k-1 = \Delta_{k-1} U(n-k) - \Delta_{k-1} U(n)$, and by Proposition 2.2, $k-1 = \Delta_{k-1} U(n-k)$. By Corollary 2.5, (5.1), and the properties of the difference operator (respectively), $0 = \sum_{i=k+1}^{2k} \Delta_{k-1} U(n-i) = \Delta_{k-1} T(n-k-1) = \Delta_{k-3} T(n-k-1) + \Delta_2 T(n-2k+2)$, both of whose last terms are thus zero, while by (1.6), $k-1 \ge \Delta_{k-1} T(n-k+1) \ge \Delta_{k-1} U(n-k) = k-1$.

But then $\Delta_2 T(n-k+1) = \Delta_{k-1} T(n-k+1) - \Delta_{k-3} T(n-k-1) = k-1$, a contradiction. Thus, $\Delta_2 T(n) \in \{0, k-1\}$ and the induction is complete. □

The polypaeon structure can be viewed as arising from the difference identity

$$\Delta_{k+1} \Delta_{k-1} U(n) = \Delta_{k-1} \Delta_{k+1} U(n). \tag{5.4}$$

This elementary difference equation causes periodicity with period $k \pm 1$ in the sequences $\Delta_{k \mp 1} U(n)$ to mutually reinforce each other: $\Delta_{k-1} U(n)$ is periodic with period $k+1$ on an interval (i.e., the left-hand side of (5.4) is zero) iff $\Delta_{k+1} U(n)$ is periodic on that interval with period $k-1$ (i.e., the right-hand side of (5.4) is zero).

Recall Note 2.1, which lists values of $\Delta_6 U(n)$ when $k = 7$. We can now recognize the values beginning with $n = 15$ as forming three paeons, then a hypercatalectic, then seven more paeons. $\Delta_6 U(n)$ is therefore periodic with period 8 on the interval from $n = 15$ to $n = 38$, and again from $n = 40$ to $n = 95$. This means that according to (5.4), $\Delta_8 U(n)$, whatever its values may be, is periodic with period 6 on those two ranges of n. The reader may verify that in fact $\Delta_8 U(n) = 0$, except at the beginning of each period, when it is equal to 6.

For general odd k, we will show that the periods in $\Delta_{k+1}U(n)$ behave similarly to the case $k = 7$: the difference sequence is 0 except at the beginning of each period of length $k-1$, where it has the value $k-1$. So we have two difference sequences, $\Delta_{k-1}U(n)$ and $\Delta_{k+1}U(n)$, which are mostly zero but almost periodically equal to $k-1$ (with different periods). In what follows, we will need to discuss the phase difference between these sequences, which we now define.

DEFINITION 5.3. *Let n mark the beginning of a paeon in $\Delta_{k-1}U(n)$, that is, $\varphi(n) = P$. The* phase difference $\theta(n)$ *between the sequences $\Delta_{k\pm1}U(n)$ is the smallest nonnegative value for which $\Delta_{k+1}U(n + \theta(n)) \neq 0$.*

Since the periods in $\Delta_{k+1}U$ are shorter than the periods in $\Delta_{k-1}U$ by 2, the phase difference $\theta(n)$ decreases by 2 (modulo $k-1$) with each successive consecutive paeon in $\Delta_{k-1}U$. We will also find that because of the way in which the periodicity in the two difference sequences is mutually reinforcing, the only time that a hypercatalectic can occur in $\Delta_{k-1}U$ is when the phase difference is about to drop from 2 to 0, $\theta(n)$ returns to zero at the beginning of each polypaeon and $\Delta_{k+1}U(n)$ is zero except where it is $k-1$. We will use these ideas in the following proof of the polypaeon structure of $\Delta_{k-1}U(n)$, which while lengthy illustrates well the added complexity of the case of general odd k.

PROPOSITION 5.4 (polypaeon structure of $\Delta_{k-1}U(n)$).

(I) *Along each polypaeon, $\Delta_{k+1}U$ consists of $\frac{k+1}{2}$ copies of the $k-1$ integers $(k-1, 0, \ldots, 0)$. Formally, suppose $\varphi(n) = P$ marks the start of a polypaeon, and $n \leq m < n + \frac{k^2-1}{2}$. Then $\Delta_{k+1}U(m) = k-1$ if $m \equiv n \pmod{k-1}$, and $\Delta_{k+1}U(m) = 0$ otherwise.*

(II) *At a hypercatalectic, $\Delta_{k+1}U$ vanishes. Formally, if $\varphi(n) = H$, then we have $\Delta_{k+1}U(n) = 0$.*

(III) *Polypaeons are followed by either polypaeons or hypercatalectics. Formally, if $\varphi(n - \frac{k^2-1}{2}) = P$ marks the start of a polypaeon, then that polypaeon is followed at n by either another polypaeon or a hypercatalectic.*

(IV) *Hypercatalectics are followed by polypaeons. Formally, if $\varphi(n-1) = H$, then a polypaeon starts at n.*

Proof. It is laborious, but not difficult, to verify directly that there is a polypaeon which starts at $2k+1$ and is followed by a hypercatalectic at $\frac{1}{2}(k^2 + 4k + 1)$, along both of which $\Delta_{k+1}U$ has the required values.

We proceed by induction, assuming that all four statements hold for all lesser n. In what follows, we will use statements (I) and (II) to compute values of $\Delta_{k+1}U$ that precede n, and use statements (III), (IV) and the above-mentioned presence of the initial polypaeon starting at $2k+1$ and hypercatalectic at $\frac{1}{2}k^2 + 4k + 1$ to ensure the polypaeon-hypercatalectic structure that immediately precedes n. We begin by proving statement (I).

By (5.4) on the commutativity of the difference operator, $\Delta_{k+1}U(n) = \Delta_{k+1}U(n - k + 1) + \Delta_{k-1}U(n) - \Delta_{k-1}U(n-k-1)$. By induction, $\Delta_{k+1}U(n-k+1) = \Delta_{k-1}U(n - k - 1)$, with both being equal to zero if n is preceded by a hypercatalectic and equal to $k-1$ if n is preceded by a paeon. $\Delta_{k-1}U(n) = k-1$ by assumption, so $\Delta_{k+1}U(n)$ is also equal to $k-1$. The rest of statement (I) follows by induction (to obtain earlier values of $\Delta_{k+1}U$) and the periodicity that (5.4) and the repeated paeons in the polypaeon impose on $\Delta_{k+1}U$.

Proving statement (II) simply consists of using (5.4) and induction to compute $\Delta_{k+1}U(n) = \Delta_{k+1}U(n-k+1) + \Delta_{k-1}U(n) - \Delta_{k-1}U(n-k-1) = 0+0-0 = 0$.

Statement (III) is more difficult to prove. It follows directly from Corollary 2.5 and Definition 5.1 that what follows the last paeon of a polypaeon must be either a hypercatalectic (if $\Delta_{k-1}U(n) = 0$) or a paeon (if $\Delta_{k-1}U(n) = k-1$). We assume therefore the latter case. It is not obvious at the outset that the paeon at n is the first paeon of a full polypaeon. We will prove this in fact true by showing (i) that $\theta(n) = 0$, (ii) that θ drops by 2 modulo $k-1$ with each successive paeon, and (iii) that the only time the next hypercatalectic can occur is when θ would drop from 2 to 0.

In our proof above of statement (I), we showed that $\varphi(n) = P$ implies that $\Delta_{k+1}U(n) = k-1$, and showed how to use periodicity to calculate the values of $\Delta_{k+1}U(n)$ that are induced by consecutive paeons, showing (ii). Since $\Delta_{k+1}U(n) = k-1$, we have $\theta(n) = 0$. And since the $(k-1)$'s in $\Delta_{k+1}U$ recur with period $k-1$ as long as we continue to have consecutive paeons at $n' \in \{n, n+k+1, n+2(k+1), \dots\}$, then $\theta(n')$ drops by 2 modulo $k-1$ with each successive paeon.

To obtain (iii), we need to determine whether the next foot after the paeon at some n' is a paeon or a hypercatalectic. We calculate

$$
\begin{aligned}
\Delta_{k-1}U(n'+k+1) &= 0+\cdots+0+\Delta_{k-1}U(n'+k+1) \\
&= \sum_{i=0}^{k-1} \Delta_{k-1}U(n'+k+1-i) \\
&= \Delta_{k-1}T(n'+k+1) \\
&= \sum_{0\le j\ even<k-1} \Delta_2 T(n'+k+1-j) \quad \text{by (5.2).}
\end{aligned} \tag{5.5}
$$

If $\theta(n') > 2$, then $\Delta_2T(n'+\theta(n'))$ is one of these summands, which we can then compute as follows:

$$
\begin{aligned}
\Delta_2T(n'+\theta(n')) &= \sum_{i=0}^{k-1} \Delta_2U(n'+\theta(n')-i), \text{ which telescopes to} \\
&= \Delta_{k+1}U(n'+\theta(n')) + \Delta_{k-1}U(n'+\theta(n')-1) \\
&= (k-1)+0 = k-1.
\end{aligned} \tag{5.6}
$$

It is worth noting in passing that (5.6) shows that $\Delta_2T(n') = k-1$ whenever $\Delta_{k+1}U(n') = k-1$.

We now consider three cases separately: $\theta(n') > 2$, $\theta(n') = 0$, and $\theta(n') = 2$.

In the first case, if $\theta(n') > 2$, then (5.6) shows that $\Delta_{k-1}U(n'+k+1) = k-1$ starts a paeon.

In the second case, if $\theta(n') = 0$, then by the same argument used in the proof of (I) above, $\Delta_{k+1}U(n'+k-1) = k-1$. And by the same argument used in (5.6), substituting $k-1$ for $\theta(n')$, we have $\Delta_2T(n'+k-1) = k-1$. This value of Δ_2T is among the summands in the last line of (5.5), so $\Delta_{k-1}U(n'+k+1) = k-1$ begins a paeon.

It is therefore only in the third case, $\theta(n') = 2$, that a hypercatalectic can occur. This completes the proof of statement (III). All that remains now is statement (IV).

We know from Proposition 2.2 that $\Delta_{k-1}U(n)$ is either 0 or $k-1$. If $\Delta_{k-1}U(n) = k-1$, then we can use the preceding proof of statement (III) to show that it marks the beginning of a polypaeon. What we need to show then is simply that $\Delta_{k-1}U(n) \neq 0$.

To do so, we generalize the proof of Proposition 3.3, relying on the polypaeon property to ensure that the number of preceding paeons is a multiple of $\frac{k-1}{2}$, so that we can compute some key differences of T.

By induction and the presence of a first hypercatalectic at $n = \frac{1}{2}k^2 + 4k + 1$, we know that the hypercatalectic at $n-1$ is preceded by some positive number of polypaeons, the first of which is itself preceded by a hypercatalectic. For some positive $q = r\,\frac{k-1}{2}$ then, $\Delta_{k-1}U(n-k-2) = \Delta_{k-1}U(n-2(k+1)-1) = \cdots = \Delta_{k-1}U(n-q(k+1)-1) = \Delta_{k-1}U(n-(q+1)(k+1)-2) = k-1$ and $\Delta_{k-1}U(i) = 0$ for all other i in the interval $[n-(q+1)(k+1)-2, n-1]$.

Use these known values of $\Delta_{k-1}U$, the distributivity of the difference operator over the definition of T in (5.1), and the equation $\Delta_d R(n) = d - \Delta_d T(n-1)$ to compute four important differences of R:

$$
\begin{aligned}
\Delta_2 R(n) &= 2 - \Delta_2 T(n-1) \\
&= 2 - \Delta_{k+1}T(n-1) + \Delta_{k-1}T(n-3) \\
&= 2 - \sum_{i=0}^{k-1} \Delta_{k+1}U(n-1-i) + \sum_{i=0}^{k-1} \Delta_{k-1}U(n-3-i) \\
&= 2 - (k-1) + (k-1) = 2.
\end{aligned} \tag{5.7}
$$

The preceding sums are evaluated by using induction to find the values of the differences of U and observing that exactly one summand in each range is nonzero. The following three equations are derived similarly, using in addition a parity argument for (5.10):

$$\Delta_{k-1}R(n) = k-1, \tag{5.8}$$

$$\Delta_{k-1}R(n-q(k+1)-2) = k-1, \tag{5.9}$$

$$\Delta_{k-1}R(n-i(k-1)-2) = 0 \text{ for } 0 \le i \le r\,\frac{k+1}{2}. \tag{5.10}$$

By definition, $\Delta_{k-1}U(n) = U(n) - U(n-k+1) = T(R(n)) - T(R(n-k+1))$. Since $\Delta_{k-1}R(n) = k-1$, we can continue the equation $T(R(n)) - T(R(n-k+1)) = T(R(n)) - T(R(n)-k+1) = (\Delta_{k-1}T)(R(n)) = \sum_{i=0}^{k-1}(\Delta_{k-1}U)(R(n)-i)$. We need to show that this last sum is equal to $k-1$. We claim that this is so because one of its first two terms is $k-1$ and the rest are all zero. By the inductive assumption that the sequence preceding n consists of feet, it suffices to show that $(\Delta_{k-1}U)(R(n)-i) = 0$ for $i \in [2, k+1]$, forcing one of the two following differences ($i = 0$ or $i = 1$) of U to be the nonzero start of a paeon:

$$
\begin{aligned}
\sum_{i=2}^{k+1}(\Delta_{k-1}U)(R(n)-i) &= (\Delta_{k-1}T)(R(n)-2) \\
&= (\Delta_{k-1}T)(R(n-2)) \text{ by (5.7)} \\
&= (\Delta_{k-1}T)\left(R(n - r\,\frac{k+1}{2}(k-1)-2)\right) \text{ by (5.10)} \\
&= (\Delta_{k-1}T)(R(n-q(k+1)-2)) \\
&= (\Delta_{k-1}U)(n-q(k+1)-2) \text{ by (5.9)} \\
&= 0.
\end{aligned}
$$

TABLE 5.1
Function values over a paeon for odd k.

n	$T(n)$	$\Delta_{k-1}T(n)$	$R(n)$	$\Delta_{k-1}R(n)$	$\Delta_{k-1}U(n)$	$\varphi(n)$
n_0-k	$t_{k-2}-k+1$	$k-1$	?	0	0	–
n_0-k+1	t_0-k+1	$k-1$	r_0-k+1	0	0	–
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n_0-3	$t_{k-4}-k+1$	$k-1$	?	0	0	–
n_0-2	$t_{k-3}-k+1$	?	r_0-2	0	0	–
n_0-1	$t_{k-2}-k+1$	0	r_1	?	0	?
$\boxed{n_0}$	$\boxed{t_0}$	$k-1$	$\boxed{r_0}$	$k-1$	$k-1$	P
n_0+1	$\boxed{t_1}$	$k-1$	$\boxed{r_1}$	0	0	–
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n_0+k-2	$\boxed{t_{k-2}}$	$k-1$	$\boxed{r_{k-2}}$	0	0	–
n_0+k-1	t_0+k-1	$k-1$	r_0	0	0	–
n_0+k	t_1	0	r_1	0	0	–
n_0+k+1	t_2+d	d	r_2+k-1	$k-1$	$\boxed{d}$	P or H
n_0+k+2	t_3+k-1	$k-1$	r_3+ $k-1-d$	$k-1-d$	$k-1-d$	– or P
n_0+k+3	t_4+k-1	$k-1$	r_4	0	0	–

All the summands are therefore zero, and so one of $(\Delta_{k-1}U)(R(n)-1)$ and $(\Delta_{k-1}U)(R(n))$ must be $k-1$ and we are done. ∎

The fundamental result that $\Delta_{k-1}U$ consists of polypaeons separated by at most one zero (constituting a hypercatalectic) now follows as a direct consequence.

COROLLARY 5.5 (a generalized version of Proposition 3.3). $\{\Delta_{k-1}U(n)\}_{n=2k+1}^{\infty}$ *consists only of feet, and each of its paeons occurs within a run of polypaeons.*

Proof. The proof follows immediately from Proposition 5.4. ∎

Since the remainder of the results in this section may be proven by straightforward generalizations of the corresponding earlier propositions, we state them here without proof.

THEOREM 5.6 (a generalized version of Foot Pattern Theorem 3.4). *Suppose the parameters $n_0, t_0, \dots, t_{k-2}, r_0, \dots, r_{k-2}$ and d satisfy all of the following conditions: $n_0 \geq 2k+1$, $\Delta_{k-1}U(n_0) = k-1$, $\Delta_{k-1}U(n_0+k+1) = d$, and for $0 \leq i \leq k-2$, $T(n_0+i) = t_i$, and $R(n_0+i) = r_i$. Then $T(n)$, $\Delta_{k-1}T(n)$, $R(n)$, $\Delta_{k-1}R(n)$, $\Delta_{k-1}U(n)$, and $\varphi(n)$ have the values shown in Table* 5.1.

DEFINITION 5.7 (generation). *For any $g > 0$, let $m_g := \frac{1}{k-1}(k^{g+1}+k^2-k-1) = k + \sum_{i=0}^{g} k^i$ and call the interval $[m_g, m_{g+1}-1]$ the* gth generation, *written as* gen(g). *We partition the gth generation into two nonconsecutive subsequences: the* gth even semigeneration $\mathrm{sg}_0(g) := \{n \in \mathrm{gen}(g) \mid n \equiv m_g \pmod 2\}$ *and the* gth odd semigeneration $\mathrm{sg}_1(g) := \{n \in \mathrm{gen}(g) \mid n \not\equiv m_g \pmod 2\}$. *For any sequence $s(n)$, we will refer to the subsequence $\{s(n) \mid n \in \mathrm{gen}(g),\ s(n)$ defined$\}$ as the* gth generation of s *and similarly for semigenerations. An* even (odd) *foot is one that starts in an even (odd) semigeneration. Note that because the length k^{g+1} of* gen(g) *is odd, m_g and m_{g+1} always have opposite parity.*

THEOREM 5.8 (a generalized version of Generation Pattern Theorem 3.15). *The gth generation of $\Delta_{k-1}U$ consists entirely of k^g feet, which make up $k^{g-1}+1$ lines.*

Its last line consists only of (odd) paeons, the last of which has Foot Pattern Theorem 5.6 parameters $n_0 = m_{g+1} - k - 1$, $t_0 = k^{g+1} - k + 1$, $t_1 = \cdots = t_{k-2} = k^{g+1}$, $r_0 = m_g - 2$, $r_1 = m_g - 1$ *and for* $2 \le i \le k-2$, $r_i = m_g + i - 2$.

At this point, we have generalized all of section 3 for odd k, describing the generational structure of the sequences. The remaining results in this section generalize the indicated propositions and theorems about properties of the sequences, proven for $k = 3$ in section 4.

PROPOSITION 5.9 (a generalized version of Proposition 4.10). *The sequence consisting of the number of paeons in each even line in the* $(g+1)$*st generation of* $\Delta_2 U$ *is*

$$\left\{ \frac{k+1}{2}\, q_{0,g}, \overbrace{\frac{k-1}{2}, \ldots, \frac{k-1}{2}}^{q_{1,g}}, \ldots, \frac{k+1}{2}\, q_{k^{g-1}-1,g}, \overbrace{\frac{k-1}{2}, \ldots, \frac{k-1}{2}}^{q_{k^{g-1},g}} \right\}.$$

THEOREM 5.10 (a generalized version of Theorem 4.11). *Each generation of the sequence* $\Delta_{k-1}U(n)$ *consists of a palindromic sequence of feet.*

PROPOSITION 5.11 (a generalized version of Proposition 4.13). $q_{i,g}$ *is always* $\frac{k-1}{2}$ *times a power of* $\frac{k+1}{2}$.

PROPOSITION 5.12 (a generalized version of Proposition 4.14). *For positive* g *and* $0 \le x \le m_{g+1} - k - 1$, *the sum* $T(m_g + x) + T(m_{g+1} - k - 1 - x) = k^g + k^{g+1}$ *is constant. For* $0 \le y \le m_{g+1} - 2k$, *the sum* $U(m_g + y) + U(m_{g+1} - 2k - y) = k^{g-1} + k^g$ *is constant.*

PROPOSITION 5.13 (a generalized version of Proposition 4.15). *The mean value of the gth generation of* $T(n)$ *is* $\frac{1}{2}(k^{g+1} + k^g + k - 1)$.

COROLLARY 5.14 (a generalized version of Corollary 4.16). *The asymptotic value of* $\frac{T(n)}{n}$ *is* $\frac{k-1}{k}$.

PROPOSITION 5.15 (a generalized version of Proposition 4.17). *The mean value of the gth generation of* $\Delta_2 T(n)$ *is* $\frac{2(k-1)}{k}$. *The mean value of the gth generation of* $\Delta_2 U(n)$ *is* $\frac{2(k-1)}{k^2}$.

We conclude this section with a few conjectures based on empirical evidence. This paper does not for the most part discuss the case of $a \neq 0$, but the following observation seems closely enough related to the palindromicity property proven in section 4 to be worth mentioning here.

CONJECTURE 5.16 (a generalized version of Proposition 4.11). *For general* a *and odd* k, $\Delta_{k-1}U$ *has a palindromic generational structure, but there are* k *extra zeros after each generation.*

We have observed the following curious property, which describes a surprising way in which the even and odd q sequences dovetail together.

CONJECTURE 5.17. *For* $a = 0$, *odd* k, *and sufficiently large* g, *the number of consecutive times that* $q_{i,g}$ *has the value* $\frac{k-1}{2}$ *is always* k.

As in the case of $k = 3$, maximal runs of identical values in the $T(n)$ sequence tend to occur at the ends of generations. This appears to be true for any odd k.

CONJECTURE 5.18 (a generalized version of Theorem 4.3). *For* $a = 0$ *and odd* k, $T(n-k) = \cdots = T(n-1)$ *iff* $n = m_g$ *for some* g.

We conclude this section with two open questions suggesting avenues for further research, and welcome correspondence concerning them.

QUESTION 5.19. *What can be said of the sequences generated by other initial values? In particular, which initial values give sequences which are well defined, and* *which ones lead to a generational structure of the sort described in this paper?*

QUESTION 5.20. *Let $a = 0$ and k be odd. The palindromic symmetry property means that the values of each sequence at odd integers can be simply expressed in terms of the values at even integers, and vice versa. Is there a simple recurrence for $T(2n)$ in terms of T at other even integers?*

6. Conjectures for even *k*. Figure 6.1 shows $T_{0,4}(n)$ with the usual initial values $(1, 1, 1, 1)$. Unlike when k is odd, we do not see a bifurcation into two intertwined subsequences; rather, the sequence stays close to the expected line $T = \frac{3}{4}n$.

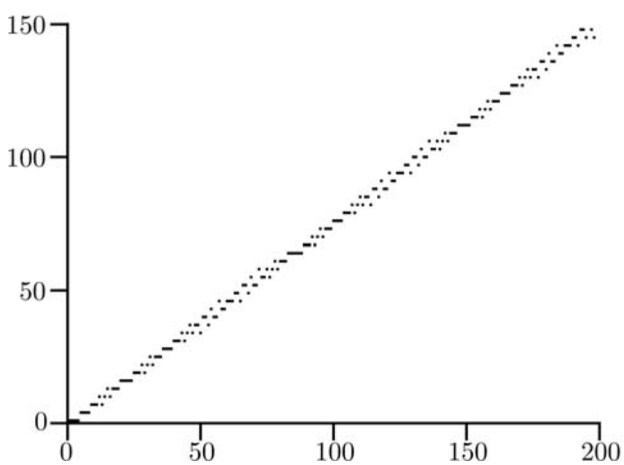
Nonetheless, because of Corollary 2.5, we know that the sequence $\Delta_3 U_{0,4}(n)$ has a block structure of runs of paeons separated by one or more zeros (resembling hypercatalectics). These zeros can appear singly or (unlike when k is odd) multiply: there are two zeros in a row at $n = 87$ and $n = 88$ following the paeon that runs from $n = 82$ to $n = 86$. Still, if we make a minor change in our definition of feet to allow for consecutive hypercatalectics, we can define feet, lines, and generations in a natural way that fits our empirical observations.

DEFINITION 6.1. *Let $a = 0$ and k be even. A* paeon *is a sequence*

$$\{k-1, \overbrace{0, \dots, 0}^{k}\}$$

of $k + 1$ consecutive values of $\Delta_{k-1}U(n)$. A hypercatalectic *is a singleton sequence $\{0\}$ in $\{\Delta_{k-1}U(n)\}$ that is not part of a paeon. A* foot *is either a paeon or a hypercatalectic. For convenience, we will write* {P} *interchangeably with the paeon $\{k-1, 0, \dots, 0\}$ and likewise $\{H\}$ with the hypercatalectic $\{0\}$ when listing values of $\Delta_{k-1}U$. We also define $\varphi(n)$ as the symbol P if $\Delta_{k-1}U(n) = k-1$ begins a paeon, H if $\Delta_{k-1}U(n) = 0$ is a hypercatalectic, and leave it undefined otherwise. For any $g > 0$, let $m_g := \frac{1}{k-1}(k^{g+1} + k^2 - k - 1) = k + \sum_{i=0}^{g} k^i$ and call the interval $[m_g, m_{g+1} - 1]$ the* gth generation*, written as* gen(g)*. For any sequence $s(n)$, we will refer to the subsequence $\{s(n) \mid n \in \text{gen}(g),\ s(n)\ \text{defined}\}$ as the* gth generation of s *and similarly for semigenerations. An* even (odd) *foot is one that starts in an even (odd) semigeneration.*

We say that this definition is a natural one, because it allows some of the sequence



n	$T(n)$	n	$T(n)$
1	1	13	7
2	1	14	10
3	1	15	13
4	1	16	10
5	4	17	13
6	4	18	13
7	4	19	13
8	4	20	16
9	7	21	16
10	7	22	16
11	7	23	16
12	10	24	16

FIG. 6.1. *$T_{0,4}(n)$ with initial values $(1, 1, 1, 1)$.*

properties we observed earlier for odd k to continue to hold for even k. For example, there is always a run of equal values of $T(n)$ at the end of each generation, as can be seen in Figure 6.1 preceding $n = m_2 = 25$ and $n = m_3 = 89$.

In light of this definition, it is instructive to look back at the case $k = 2$ (Conolly's sequence), which is completely understood, in light of this definition. In its corresponding $\Delta_1 U$ sequence, beginning with the second generation at $n = m_2 = 9$, paeons always occur in pairs, and the number of consecutive hypercatalectics that occur between pairs of paeons forms the previously mentioned Gray binary sequence $1, 2, 1, 3, 1, 2, 1, 4, \ldots$, with an extra hypercatalectic at the end of each generation.

We believe that all of section 3 can be generalized to even k. We do not, however, observe palindromic symmetry or polypaeon structure, so not all of section 4 can be carried over to this case. For example, when $k = 4$, the first generation of $\Delta_2 U$ consists of the feet $\{P, P, P, H\}$; the second generation is $\{P, P, P, P, P, H, P, P, P, P, P, H, P, P, H, H\}$; the third generation begins with a copy of the second generation, includes a run of eight paeons, and ends with three consecutive hypercatalectics. None of the generations are palindromic, and consecutive paeons appear in varying and relatively prime numbers.

The following four conjectures state how we believe our results of section 3 will generalize to even k, based on empirical evidence.

Conjecture 6.2. *For $a = 0$, even k, and the usual initial conditions $T_{0,k}(n) = 1$ for $1 \le n \le k$, implicitly define $f(R(n)) := n$ for $n \in \operatorname{Dom}\varphi$. Then f is well defined on its domain and $f(r)$ is the least n for which $r = R(n)$.*

Conjecture 6.3 (a generalized version of Foot Pattern Theorem 5.6). *Let $a = 0$, k be even, and assume the usual initial conditions $T_{0,k}(n) = 1$ for $1 \le n \le k$. Suppose the parameters $n_0, t_0, \ldots, t_{k-2}, r_0, \ldots, r_{k-2}$ and d satisfy all of the following conditions: $n_0 \ge 2k + 1$, $\Delta_{k-1}U(n_0) = k - 1$, $\Delta_{k-1}U(n_0 + k + 1) = d$, and for $0 \le i \le k - 2$, $T(n_0 + i) = t_i$, and $R(n_0 + i) = r_i$. Then $T(n)$, $\Delta_{k-1}T(n)$, $R(n)$, $\Delta_{k-1}R(n)$, $\Delta_{k-1}U(n)$, and $\varphi(n)$ have the values shown in Table* 5.1.

Conjecture 6.4 (a generalized version of Generational Correspondence Theorem 3.14). *Let $a = 0$ and k be even. Then the diagram*

$$\begin{array}{ccc} \operatorname{Dom} f & \underset{R}{\overset{f}{\rightleftarrows}} & \operatorname{Ran} f \\ {\scriptstyle \Delta_{k-1}T}\downarrow & & {\scriptstyle \varphi}\downarrow \\ \{0, k-1\} & \xrightarrow[k-1 \mapsto P]{0 \mapsto H} & \{H, P\} \end{array}$$

commutes. That is, for $k + 1 \le r \in \operatorname{Dom} f$, $\varphi(f(r)) = P$ iff $\Delta_{k-1}T(r) = k - 1$.

Conjecture 6.5 (a generalized version of Generation Pattern Theorem 3.15). *Let $a = 0$ and k be even. Then the gth generation of $\Delta_{k-1}U$ consists entirely of k^g feet, which make up k^{g-1} lines. The generation ends with g hypercatalectics preceded by a paeon, and if we let $n_0 := m_{g+1} - k - 1$, then for $0 \le i \le k - 2$, $T(n_0 + i) = k^{g+1}$ and $R(n_0 + i) = m_g - k - 1 + i$.*

While we do not have palindromic symmetry when k is even, the existence of f appears to be sufficient to generalize the following sequence property observed in section 5.

Conjecture 6.6 (a generalized version of Proposition 5.15). *For $a = 0$ and any (not necessarily odd) k, the mean value of the gth generation of $\Delta_2 T(n)$ is $\frac{2(k-1)}{k}$. The mean value of the gth generation of $\Delta_2 U(n)$ is $\frac{2(k-1)}{k^2}$.*

As in the case of odd k described in Conjecture 5.18, it is also true for even k that maximal runs of identical values in the $T(n)$ sequence tend to occur at the ends of generations.

CONJECTURE 6.7 (a generalized version of Theorem 4.3). *For $a = 0$, even k and any g, $T(m_{g+1} - k - g) = \cdots = T(m_{g+1} - 1)$ and there are no earlier runs of $k + g$ identical consecutive values of $T(n)$.*

In a forthcoming communication, we hope to resolve the above conjectures in the broader context of the determination of the complete structure of our sequences for even k.

REFERENCES

[1] R. B. J. T. ALLENBY AND R. C. SMITH, *Some sequences resembling Hofstadter's*, J. Korean Math. Soc., 40 (2003), pp. 921–932.

[2] ARISTOTLE, *Ars Rhetorica*, Sir David Ross, ed., Oxford University Press, Oxford, 1959, pp. 158–159, Bekker index 1409a.

[3] B.W. CONOLLY, *Meta-Fibonacci sequences*, in Fibonacci & Lucas Numbers, and the Golden Section, S. Vajda, ed., Wiley, New York, 1986, pp. 127–137.

[4] C. H. ELZINGA, *private communication*, October 2001.

[5] J. HIGHAM AND S. TANNY, *More well-behaved meta-Fibonacci sequences*, Congr. Numer., 98 (1993), pp. 3–17.

[6] J. HIGHAM AND S. TANNY, *A tamely chaotic meta-Fibonacci sequence*, Congr. Numer., 99 (1994), pp. 67–94.

[7] D. HOFSTADTER, *Gödel, Escher, Bach: An Eternal Golden Braid*, Basic Books, New York, 1979.

[8] G. HUBER AND D. HOFSTADTER, *private communication*, 1999.

[9] G. HUBER, *private communication*, May 2002.

[10] T. KUBO AND R. VAKIL, *On Conway's recursive sequence*, Discrete Math., 152 (1996), pp. 225–252.

[11] I. LANCASHIRE, *Glossary of Poetic Terms (Version* 2.07*)*, Representative Poetry Online (Version 2.11), http://www.library.utoronto.ca/utel/rp/intro.html (1994).

[12] C. L. MALLOWS, *Conway's challenge sequence*, J. Amer. Math. Mon., 98 (1991), pp. 5–20.

[13] N. J. A. SLOANE, *Sloane's Encyclopedia of Integer Sequences*, http://www.research.att.com/~njas/sequences.

[14] S. M. TANNY, *A well-behaved cousin of the Hofstadter sequence*, J. Discrete Math., 105 (1992), pp. 227–239.